Endogenous Belief Switching
Revisiting the Forward Guidance Puzzle*

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Abstract

Forward guidance has emerged as a crucial tool for central banks as short-term interest rates approach the zero lower bound. While economic theory has extensively examined the effectiveness of unconventional monetary policy, recent attention has focused on the fundamental role of expectations in macroeconomic models. However, limited research exists on forward guidance in an adaptive learning environment, particularly when expectations become adaptive at the zero lower bound. This paper aims to address this gap by investigating the forward guidance puzzle within an adaptive learning framework and emphasizing the significance of monetary policy in expectation formation.

To explain the role of learning and dynamic expectation formation in the context of unconventional monetary policy, I propose the framework of endogenous belief switching. This framework combines rational and adaptive learning approaches to solve the forward guidance puzzle. It posits that expectations are determined by central bank actions, making the effectiveness of forward guidance endogenous: I allow agents to learn the transmission of pre-announced policy rate changes based by alternating between forward-looking beliefs or focusing solely on current conditions and forming backward-looking beliefs. I endogenize belief switching by incorporating a mean squared learning transition between these two belief regimes. Agents update their beliefs every period using a switching Kálmán filter (Murphy, 1998), which allows them to dynamically determine whether to adopt a forward-looking or backward-looking perspective based on the probability that either regime best describes the economy.

Simulation results demonstrate that the effectiveness of forward guidance is nonlinear. When agents are adaptive and backward-looking, the forward guidance puzzle does not arise. However, if expectations are adaptive and forward-looking, the puzzle emerges. The framework predicts that forward guidance is highly effective in low uncertainty environments, where the model aligns well with the data and observation error is minimal. Conversely, in high uncertainty economies, forward guidance can become ineffective. In such cases, agents may opt to become backward-looking due to excessive noise relative to the signal provided by forward guidance. However, agents can learn to trust the central bank if it conveys a strong enough signal regarding its commitment.

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1 Introduction

In the pre-Great Recession era, the literature on monetary policy largely converged on the notion of setting the short-term interest rate, the inter-temporal return of assets, known as the policy rate. Extensive research has documented the transmission mechanism of surprise changes in short-term interest rates onto the economy, considering financial, nominal, and real frictions. This has been achieved through the use of VARs or DSGE models (e.g., Sims (1980), Christiano et al. (1999), Clarida et al. (1999),Christiano et al. (2005)).

While the effects of short-term interest rates in normal times is well understood, the policy rate has been constrained by the zero lower bound (ZLB) during the financial crisis in most developed economies. To overcome the disastrous theoretical consequences a constrained monetary policy meant for models, alternative tools were developed and proposed, and the term unconventional monetary policy has been coined. Major central banks have employed these new tools, such as announcements about the future path of the policy rate, forward guidance, or quantitative easing[1].

As economic theory of unconventional monetary policy has developed, so too have the interpretations of the primary channels through which these policy instruments operate. Forward guidance, in particular, emerged as a promising tool for overcoming the challenges posed by the ZLB. Research by Eggertsson and Woodford (2003) highlighted that when a central bank commits to an interest rate path lower than what would be implemented under normal circumstances, it can have an additional expansionary impact. This effect primarily operates through the credible pegging of expectations. Due to the inherent persistence of *rational expectations, this peg translates into a large impact of the announcements. As unconventional measures were further explored and implemented, so evolved our interpretation of unconventional monetary policy. For instance, recent understanding of quantitative easing attributes its effectiveness to the signaling channel, and the implicit forward guidance it conveys (Bhattarai et al., 2015). Nonetheless, concerns regarding time consistency and credible commitment to a "lower for longer" policy have been raised (Woodford, 2012) and remain a theoretical and practical concern.

Evidence suggests that expectations become de-anchored from their inflation targets and become adaptive at the ZLB. Inflation expectations are considered anchored when long-term expectations do not respond to surprises and news. However, studies (e.g.Beechey et al. (2011), Van der Cruijsen and Demertzis (2011) Dovern and Kenny (2017); Łyziak and Paloviita (2017)) have shown that inflation expectations do respond to inflation news and are endogenous to monetary policy, indicating an adaptive nature of expectations. Mario Draghi also noted

[1] Measures involving a change in the size and the composition of the central bank balance sheet.

that the ”risk that too prolonged period of low inflation becomes embedded in inflation expectations” Draghi (2014).

Therefore, it is of high relevance to understand how forward guidance works in an adaptive learning environment. Standard medium-scale DSGE models often overestimate the impact of forward guidance on the macroeconomy, leading to the ”forward guidance puzzle.” The failure of these models to explain the missing effectiveness of forward guidance has triggered criticism of the way expectation formation is modeled. While deviations from the standard rational expectations (RE) benchmark have been shown to mitigate or overcome the forward guidance puzzle, there is still room for exploration. This paper fills that gap by proposing a novel adaptive learning framework in which expectation formation is endogenously switching. The framework has also policy relevance, as it enables to study of the evolution of central bank credibility in light of forward guidance policies.

The main contributions of this paper are as follows: First, it demonstrates that constant gain adaptive learning can overcome the forward guidance puzzle. It distinguishes between types of constant gain adaptive expectations, backward and forward-looking beliefs. The paper models forward guidance under both backward and forward-looking adaptive expectations, showing that the forward guidance puzzle does not emerge when expectations are backward-looking. However, forward-looking constant gain adaptive expectations still exhibit the forward guidance puzzle.

Second, building on these insights, the paper proposes an endogenous belief switching framework, regime switching between backward and forward-looking expectations. This framework allows for endogenous efficacy of forward guidance and provides a simple way to model how central banks can endogenously determine the share of attentive agents, and with it influence their credibility and effectiveness of forward guidance.

Third, using this framework, the paper investigates the dynamic evolution of central bank credibility and highlights the conditions under which announcing forward guidance can enhance or erode central bank credibility. The endogenous belief switching framework provides a policy-relevant interpretation of the prolonged periods near the zero lower bound observed in multiple countries, suggesting that central banks can be partially responsible for the severity of the liquidity trap by failing to provide strong enough signals to overcome heightened uncertainty, thus endogenously leading to backward-looking expectations. Furthermore, the framework implies that close to the zero lower bound, there is limited room for monetary policy to provide strong contemporaneous accommodation necessary for maintaining credibility, potentially rendering forward guidance less effective. These findings call for a rethinking of central bank communication policies, as the model predicts that the fit of the central bank’s communicated model to the data and the understanding of its policy function are crucial for maintaining credibility.

In summary, this paper develops a novel form of expectations, referred to as endogenous belief switching, and highlights the role of learning and dynamic expectation formation in the context of unconventional monetary policy.

1.1 Forward Guidance Puzzle

Forward guidance operates on the premise that agents respond to credible announcements about future policy shocks. In this paper, I consider time-dependent forward guidance[2], where the central bank announces and delivers a pre-announced low level of the short-term interest rate, possibly reaching the ZLB, for a predetermined horizon.

The forward guidance puzzle gained attention after the seminal paper Del Negro et al. (2012). They showed that standard DSGE models generate excessive responses to announcements of credibly committed future interest rate changes. Carlstrom et al. (2015) expanded on the puzzle, highlighting that inflation indexation in DSGE models leads to counterintuitive policy counterintuitive reversals[3]. Therefore the forward guidance puzzle not only involves the increasing marginal impact of future announcements but also the inversion of its impact and the resulting reversals generating instability of the economy. In this paper, I primarily focus on the first aspect of the puzzle, as the second is conditional on the first.

Various proposals have been put forward to address the forward guidance puzzle. McKay et al. (2016) showed that when agents face uninsurable income risk and borrowing constraints, the precautionary savings effect mitigates the puzzle. They highlighted the sensitivity of the power of forward guidance to the assumptions about complete markets. argued that forward guidance at the ZLB has no real economic effects but significantly impacts risk premia, serving as insurance against the liquidity trap rather than a policy instrument of credible commitment.

1.2 Departures from Rational Expectations

Assumptions about how expectations are formed are central to the efficacy of forward guidance, and their role has been only recently appreciated in the literature. An alternative remedy to the puzzle lies in departing from RE. My paper is closely related to this literature. Carlstrom et al. (2015) noted that sticky information models can mitigate the forward guidance puzzle by eliminating reversals. These models assume that agents update their beliefs in a staggered manner based on an exogenous information revelation process. Sticky-information models, on the other hand, do not exhibit the forward guidance puzzle as expectations are driven by past expectations of current conditions rather than future conditions. However, the limitation of sticky-information models lies in their inherent time-dependent information revelation process, as belief formation is not state-dependent. My framework proposes to overcome the exogeneity limitation by modeling belief formation endogenously.

Groot and Mazelis (2020) propose to use Gabaix (2020)'s behavioral discounting to mitigate the forward guidance puzzle. Note, that myopia can only dampen and not resolve the puzzle.

[2] I do not consider state-contingent forward guidance. For recent discussion of the different effectiveness of types of forward guidance impact see Ehrmann et al. (2019).

[3] Policy reversals after a forward guidance mean that the impact of a policy peg can switch from highly expansionary to highly contractionary for modest changes in the length of the interest rate peg. For example, an interest rate peg of 7 periods may be strongly expansionary, but a peg of 8 periods may be sharply contractionary.

Chung et al. (2015) explored the implications of information stickiness in DSGE models and its relationship to price stickiness and the forward guidance puzzle. They showed that under sticky prices and partial indexation, forward guidance creates the puzzle due to the forward-looking nature of inflation. Inflation indexation introduces inertia, whereby lagged inflation becomes an endogenous state variable in the Phillips curve, leading to amplification effects. They show that "a credible promise to remain highly accommodative can lead to substantial effects on real activity and inflation. Under sticky prices [and partial indexation], these effects can be very large [...] , a promise of prolonged future accommodation raises future inflation, which leads to higher current inflation, which lowers real interest rates and raises output (which then raises inflation further)." (Chung et al., 2015)[p. 35]

In my model, which treats expectation formation as a signal-processing problem akin to rational inattention, I draw inspiration from the rational inattention literature pioneered by Sims (2006); Mackowiak and Wiederholt (2009); Maćkowiak and Wiederholt (2015). Rational inattention provides a micro foundation for decision-making and addresses how agents optimally allocate limited attention among signals, resulting in the best allocation to accurate signals. As mean squared adaptive learning agents also exhibit this property, rational inattention can be used to micro-found adaptive learning and endogenous beliefs, as demonstrated by Molavi (2019).

Andrade et al. (2019) explored the impact of information heterogeneity on the effectiveness of forward guidance. They proposed a model with heterogeneous interpretations of forward guidance policy and found that disagreements about the actual policy conducted can arise due to the unobservable commitment ability of central banks. Two type of beliefs emerge in response to forward guidance, "Odyssean agents - who believe in the commitment ability of the central bank - see the announcement as including some periods of extra accommodation, contingent to any possible realization of the length of the trap. By contrast, Delphic agents - who do not believe in the commitment ability - consider there will be no period of extra accommodation at the end of the trap." (Andrade et al., 2019, p.3.) They show that forward guidance will have counteracting forces on central bank commitment. In response to an Odyssean announcement Delphic agents become excessively pessimistic[4], believing the zero lower bound trap is longer than actual, that in turn may lead a central bank to find it optimal to abandon commitment and engage in a Delphic forward guidance. Their model studies the central bank's optimal reaction conditional on exogenous shares of heterogeneous beliefs.

In my work, I also consider the distinction between forward-looking and backward-looking adaptive beliefs. However, unlike existing literature, I endogenize belief formation with regime switching adaptive learning, allowing the central bank to influence its own credibility. Furthermore, I assume that both beliefs feature adaptive learning and are not rational expectations, offering a novel perspective on central bank credibility in light of forward guidance announce-

[4]And if a "sufficiently large fraction of agents are Delphic, additional periods of policy accommodation will have increasingly negative effects on current macroeconomic conditions." (Andrade et al., 2019, p.3.)

ments.

1.3 Adaptive Expectations

My paper is also based on the literature of adaptive learning. Throughout the paper I will explore two types of adaptive beliefs: backward and forward-looking beliefs. Section 2 reviews constant gain adaptive learning in detail, in this section I focus on my contribution in contrast to the literature.

I define backward-looking adaptive beliefs, where expectations are formed based on past state variables and current exogenous shocks and do not incorporate information about announced, anticipated future policy action. Translating backward-looking beliefs to forward guidance means that agents do not believe the forward guidance, from their perspective it is not credible. In other words, agents are Delphic, or inattentive to forward guidance. Section 3 discusses how to model backward-looking adaptive expectations in DSGE models[5].

On the other hand, forward-looking beliefs incorporate past state variables, current shocks, and anticipated future shocks, reflecting the credible commitment of forward guidance. It is essential to recognize that forward guidance's effectiveness relies on agents believing in it, leading to self-fulfilling expectations. Forward-looking adaptive beliefs can be seen as conditional forecasts given the announced forward guidance. Section 4 presents the modeling of forward-looking adaptive expectations.

Empirical evidence indicates the adaptive nature of expectations at the ZLB, raising the question of which type of adaptive expectations prevails in reality: backward-looking, forward-looking, or a mixture of both. In this paper, I develop a theoretical framework where the mixture of extreme adaptive beliefs arises endogenously in response to monetary policy.

Ample empirical evidence supports the adaptive nature of expectations. Ehrmann (2015) document that under persistently low inflation, inflation expectations become more reliant on lagged inflation. Dovern and Kenny (2017) show that inflation expectations are influenced by the historical track record of the central bank and respond to macroeconomic developments.

Carvalho et al. (2019) use a DSGE model with regime switching to estimate the degree of de-anchoring of expectations at the ZLB, finding that a constant gain adaptive learning structure better describes expectations than rational expectations.Mitra et al. (2012) study the effects of fiscal policy guidance in an adaptive learning environment, highlighting the different effects depending on expectation formation. Wieland (2008) introduces adaptive learning and endogenous indexation in the New-Keynesian Phillips curve, showing the cost-lowering effect of adaptive learning on disinflation.Eusepi and Preston (2010) explore the link between adaptive learning and central bank communication strategies, emphasizing the role of improved communication in enhancing macroeconomic stability.

[5] Throughout the paper I will use the term backward-looking adaptive expectations interchangeably to backward-looking beliefs.

Cole (2015) examines the impact of adaptive learning on forward guidance and its modeling as anticipated news shocks. However, his work assumes that all agents understand and respond to forward guidance, and the agents, despite being forward-looking, only perceive the current implementation of the path. I argue that this assumption is inconsistent, as forward-looking beliefs should re-anchor expectations to the full path every period, accounting for both contemporaneous and anticipated future shocks. By modeling beliefs with endogenous belief switching, my framework captures the dynamic response of central bank credibility to monetary policy actions, providing a tractable alternative to heterogeneous beliefs. The mean squared econometric interpretation of adaptive beliefs justifies the use of the switching Kálmán filter to update beliefs and endogenize belief formation. Consequently, my work extends adaptive learning by incorporating heterogeneous beliefs within a unified framework.

The remainder of the paper is organized as follows. Section 2 provides a review of constant gain adaptive learning. In Section 3, I introduce the Smets Wouters model (referred to as SW07 model) with constant gain adaptive learning based on Slobodyan and Wouters (2012). I explain how backward-looking beliefs can overcome the forward guidance puzzle. Section 4 discusses the implementation of anticipated news shocks in the adaptive learning framework. I demonstrate how the forward guidance puzzle arises. In Section 5, I develop the concept of endogenous belief switching using the switching Kálmán Filter (SKF). I illustrate the application of endogenous belief switching in models with one and two-dimensional state spaces. I also present a small-scale three-equation New Keynesian model that incorporates forward-looking and backward-looking beliefs with endogenous belief switching. This model is used to analyze the impact of the length and size of forward guidance on central bank credibility. Additionally, I revisit the SW07 model and emphasize the role of signal-to-noise ratio in endogenous expectation determination. Section 7 provides concluding remarks.

2 Constant Gain Adaptive Learning

Deviations from RE via learning in the real business cycles literature dates back to Kydland and Prescott (1982). Learning in macroeconomics is usually modelled as a signal extraction problem, where agents need to gather information from a noisy signal and form beliefs about the dynamic nature of the economy. Marcet and Sargent (1989) describe how agents' perceived law of motion (PLM) affects the actual law of motion (ALM) of the economy. They introduce least squares learning and derive conditions for convergence of the expectations, and show how they relate to the RE equilibrium.

Adaptive learning has gained popularity ever since, receiving its first textbook treatment by Evans and Honkapohja (2012). Bullard and Mitra (2002) evaluate how recursive learning influences monetary policy rules, while Preston (2005) studies the impact of long-term expectation determination on monetary policy design. Both approaches discuss the expectation stability (E-stability) of the model given monetary policy rules. From the policy perspective the shared conclusion is to have a policy design that delivers learnable RE equilibria in the limit. Goy et al. (2018) studies adaptive beliefs and endogenous heterogeneous expectations' response to forward guidance under regime switching using a heuristic switching model (Brock and Hommes, 1997; Hommes and Lustenhouwer, 2019). The two regimes assumed are either adaptive learning expectations or credibility believers. Switching is driven by the sum of squared forecast errors, where the central bank itself has a separate forecasting model to publish forecasts where the inflation target and RE equilibrium are satisfied. In contrast, I assume no information friction beyond adaptive expectations. In endogenous belief switching the regimes will be determined by the likelihood of the belief, given the past and present. In other words, the DSGE estimation using a Kálmán filter delivers that switching occurs not due to forecast errors, but due to the fit of the (updated) model to the past and present. Generally, a fit of a DSGE is driven by (one step ahead forecast, prediction) filtering errors, but these errors are evaluated, and weighted by the state covariance matrix. In my model, unlike Goy et al. (2018), I do not have N-period limits to expectation formation, and both the central bank and the private agents share the same forecasting rule. The central bank controls the size of the monetary and forward guidance shocks. However, agents can learn the effectiveness of forward guidance as the economic response to the central bank action makes forward-looking beliefs more likely.

In what follows I will employ the notation used in Dynare (Juillard, 2001) to introduce the structural, reduced, and state space representation of the dynamic stochastic general equilibrium (henceforth DSGE) model.

A DSGE's solution after a linearization around the steady state can be written in the structural form as follows:

$$A_0 \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + A_1 \begin{bmatrix} y_t \\ w_t \end{bmatrix} + A_2 E_t \left[y_{t+1} \right] + B\varepsilon_t = const, \tag{1}$$

where y_t stands for the endogenous state variable,e.g. output or inflation, w_t denotes the exogenous state variables, e.g. technology process, and ε_t is the realization of the exogenous shock, while A_0 is the backward solution of the DSGE, A_1 the contemporaneous response,A_2 is the forward solution given rational expectations. B captures the impact of the contemporaneous exogenous shocks on state variables, while the right-hand side $const$, constant accounts for endogenous drifts. A DSGE's RE solution can be then obtained in many ways, either using Sims' method, or Binder and Pesaran's algorithm (Binder and Pesaran, 1995). I initialize beliefs at their RE counterpart, i.e. solve for the RE equilibrium.

Given the uniqueness of the RE solution, the DSGE model can be written in the reduced form:

$$\begin{bmatrix} y_t \\ w_t \end{bmatrix} = \mu + T \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + R\varepsilon_t, \tag{2}$$

where the reduced form matrices T and R are non-linear functions of the structural parameters.

The intercept, μ, can be non-zero for observables that are not demeaned[6].

Finally, based on the reduced form, one can write the state space model representation of the DSGE, that is the reduced form, augmented with the observation equation:

$$y_t^{state} = \mathbf{F} y_{t-1}^{state} + \mathbf{w}_t, \tag{3}$$

$$X_t^{obs} = \mathbf{H} y_t^{state} + \mathbf{u}_t \tag{4}$$

Where y_t^{state} is the state vector with dynamic evolution $\mathbf{F}$, $\mathbf{w}_t$ is the exogenous state disturbance, with a covariance matrix Q_t. The state covariance matrix is sometimes also referred to as long-run variance of the states or mean squared error matrix of the states and it is equal to the square of R,i.e. $R * R'$, from the reduced form representation.

$\mathbf{H}$ is the emission matrix that selects the observable states of the model[7]. Finally, vector $\mathbf{u}_t$ is the noise of the observation equation, with covariance matrix U_t. Its relative size to the state disturbance determines the signal-to-noise ratio of the DSGE.

Under adaptive learning, as in Slobodyan and Wouters (2012), Marcet and Sargent (1989) and Evans and Honkapohja (2012), agents forecast the values of the forward variables as econometricians: replacing expectations with forecasts that are a linear function of the state variables. The adaptive learning literature calls the state space that enables agents to form expectations asymptotically equivalent to RE the minimum state variable space (MSV), which is a vector space spanned by the endogenous state variables forming the RE solution. In this paper, I abandon RE assumption and assume that agents form expectations about the forward-looking variables with the help of a linear function of state variables. Furthermore, I assume that agents can observe the MSV state and thus can learn the RE equilibrium equivalent beliefs over time. This observation is assumed for the time being to be exact, and will be relaxed in chapter 6, when discussing the relationship between signal and noise.

The equation that describes the agents' expectations is the PLM:

$$y_t^f = \alpha_{t-1} + \beta_{t-1}' \begin{bmatrix} y_{t-1} \\ w_t \end{bmatrix}. \tag{5}$$

Where $\alpha_{t-1}, \beta_{t-1}$ are representing beliefs of the means squared econometrician. The belief parameters are dated time $t-1$ to represent that expectations are formed based on predetermined states y_{t-1} and current shocks w_t and are usually collected into the belief matrix $\Phi_{t|t-1}$.

$$y_t^f = \alpha_{t-1} + \beta_{t-1}' \begin{bmatrix} y_{t-1} \\ w_t \end{bmatrix} = \Phi_{t-1} \cdot Z_t \tag{6}$$

The variables based on which expectations are formed, i.e. the regressors in the econometri-

[6]Note that the RE solution is time-invariant.

[7]This matrix will compress the state space of the DSGE into an subspace where observables are measured.

cians' model are usually denoted as $Z_t = [1, y_{t-1}, w_t]'$.

Subjective beliefs are thus time-dependent and are completely described by the information set available to the agents. Z_t is the state variables observable in period t, based on which beliefs are formed, these are previous states y_{t-1}, and current shocks w_t[8]; $\Phi_{t|t-1}$ is the belief matrix; and $R_{t|t-1}$ [9] is the beliefs about the accuracy of the states, the mean squared forecast errors before learning, i.e. updating. Once expectations are formed the stochastic shocks realize and the ALM is determined.Given the expectation errors agents update beliefs in amean-squaredd learning process.

Standard adaptive learning is an expectation formation process, that can be described as a sequence of events.(Evans and Honkapohja, 2012) The timing of the events is the following:

1. At the beginning of period t, the agents inherit the beliefs formed in the previous period: $\Phi_{t|t-1}, R_{t|t-1}$. Agents observe the shocks ε_t and the states of the previous period $[y_{t-1}, w_{t-1}]'$.

2. Agents form expectations based on their previous period beliefs $\Phi_{t|t-1}, R_{t|t-1}$ and the information set Z_t.

3. The current state is determined as the solution to the reduced form of the DSGE. The economy evolves given the reduced form solution implied by the belief matrix $\Phi_{t|t-1}$ and the mean-squared-error matrix of the states $R_{t|t-1}$. In other words, the solution of the model given beliefs constitute the ALM:

$$\begin{bmatrix} y_t \\ w_t \end{bmatrix}^{ALM} = \mu(\Phi_{t|t-1}, R_{t|t-1}) + T(\Phi_{t|t-1}, R_{t|t-1}) \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + R_{t|t-1}\varepsilon_t. \tag{7}$$

   Note that both the steady state as well as the policy function is time dependent. Both μ and T are the function of the beliefs $\Phi_{t|t-1}, R_{t-1|t}$. Furthermore, information captured in Z_t enters in two ways, first in the form of the past state variables, y_{t-1}, multiplied by the policy function $T(\Phi_{t|t-1}, R_{t|t-1})$ and second in the form of the exogenous variables w_t. The latter as, first, the impact of past exogenous variables w_{t-1}, through the policy function and, second, as the impact of current shocks through the contemporaneous response in $R_{t|t-1}\varepsilon_t$.

4. Current states $[y_t, w_t]^{ALM'}$ are then revealed. Agents update their beliefs, given the errors

[8]forward-looking agents will have to form expectations about more shocks that are in the MSV as the backward-looking agents.

[9]As you will see, forward-looking beliefs will have belief matrices that are of a larger dimension than backward-looking!

they made following the learning of equations:

$$\Phi_{t|t} = \Phi_{t|t-1} + \tau R_{t|t}^{-1} Z_t (y_t^{ALM} - y_t^f) \tag{8}$$

$$R_{t|t} = R_{t|t-1} + \tau (Z_t Z_t' - R_{t|t-1}). \tag{9}$$

I build the endogenous belief switching model around this standard, a well-understood learning framework. In what follows, I present the SW07 model with adaptive learning and discuss what forward and backward-looking beliefs imply for its solution.

In a nutshell, the Taylor rule under backward-looking beliefs will only feature i.i.d. monetary policy shocks, while forward-looking beliefs will include policy shocks that follow an integrated moving average process of the order of the forward guidance length.

Beliefs under both regimes will be spanned by the same MSV space, yet will be based on different information sets (Z_t). forward-looking agents will form expectations knowing the anticipated news shocks, i.e. based on a larger belief matrix Φ_t, considering a larger shock variance space, R_t. In the next section, I argue that the baseline SW07 did not feature anticipated news shocks and thus is consistent with backward-looking beliefs. I also show that the extended path simulation is the right approach to solve for the forward guidance when adaptive beliefs are backward-looking. Lastly, I introduce forward-looking beliefs, by incorporating anticipated news shocks.

2.1 The Smets Wouters Model with Constant Gain Adaptive Learning

The Smets and Wouters (2007) model is a medium scale DSGE model estimated for the US economy, following the work of Christiano et al. (2005) it contains both nominal an real frictions. Households maximize expected utility over an infinite horizon, given their habit formation. Labour services are aggregated by a union facing nominal Calvo wage rigidities. Households make consumption and savings decisions given investment adjustments costs. Capital faces capital utilization costs that will affects its use of intensity. Intermediate firms produce differentiated goods using labour and capital as input, and face Calvo type nominal rigidities. The both wage and product pricing is subject to partial indexation to lagged inflation. Together with marginal costs that are dependent on real wages, rental rate of capital and an exogenous technology this results in a Phillips Curve that is both backward and forward-looking. Therefore inflation dynamics have both an expectation and a lag term. Monetary policy is described by a Taylor type rule, with interest rate smoothing in the reaction to inflation- and output gap. Importantly for the current application and forward guidance, the baseline model does not feature anticipated news shocks about the future interest rate shocks, i.e. agents do not form beliefs nor have information about future monetary policy shocks. Therefore the baseline model's minimum state variable RE solution will represent the initial beliefs of the backward-looking agents. Recall, backward-looking beliefs do not consider forward guidance credible, and are inattentive, do not incorporate pre-announced future monetary policy shocks into their expectations, thus they will not respond to signals about the path, neither in size nor in length. The output gap is defined against the flex price equilibrium level of output[10].

The adaptive learning version of the SW07 model contains 44 endogenous[11] state variables, of which 13 are forward-looking[12]. For notational simplicity, I denote all the state variables y_t and collect them into a single vector.

Furthermore, the model is driven by seven shocks: The neutral and investment-specific technology shock, risk premium shock, exogenous spending shock and monetary policy shock are AR(1) processes, while price and wage mark-up disturbances are ARMA(1,1). The vector w_t collects all seven exogenous variables, as well as the lagged innovations ε_{t-1} for the mark-up shocks.

Similar to Slobodyan and Wouters (2012), I assume constant gain adaptive learning The constant gain adaptive learning can be interpreted as perpetual learning (Eusepi and Preston, 2011). The gain parameter being constant implies that the rate at which older observations are discounted follows a power function. This also means that expectations forget the past fast and pay more attention to recent forecast errors. The power of decay being fixed is in contrast to the variance-weighted approach of optimal control based, e.g. Kálmán filter learning[13]. Constant gain has advantages in terms of the interpretation of the policy function.

I assume that the information set agents use in forecasting spans the space of the baseline SW model's expectations under RE. I introduce a deviation from RE by including the constant (μ in 2 in the beliefs[14]. Similar to Slobodyan and Wouters (2012), I initiate beliefs at the policy function that is consistent with the MSV RE equilibrium solution. The deviations of the PLM from the ALM will arise due to the realization of stochastic shocks. Since expectations are formed using linear quadratic econometric models given observables and feature constant terms in them, it is instructive to check if expectations are stable, thus I discard all PLMs that are explosive or inaccurate. Due to the linear approximation of the solution around the RE steady state, the solution is only locally accurate, therefore for both stability and accuracy purposes, I only

[10]This is in contrast to the adaptive learning SW type model presented in Slobodyan and Wouters (2012), where output gap is just a deviation of output from the neutral. Their reason to drop the flex price equilibrium is to considerably reduce the number of forward variables agents have to solve.

[11]The number of endogenous state variables are larger than in the original SW model, as the Macroeconomic Model Database adds fiscal variables, and joint variables to the system, thus the model has the following state variables: a, b, c, cf, eg, epinfma, ewma, fispol, g, inflation, inflationq, inflationql, inflationql2, inflationqls, interest, inve, invef, k, kf, kp, kpf, lab, labf, mc, ms, output, outputgap, pinf, pinf4, pk, pkf, qs, r, rk, rkf, rrf, spinf, sw, w, wf, y, yf, zcap, zcapf.

[12]The forward-looking variables are the following: c, cf, inve, invef, lab, labf, pinf, inflationq, pk, pkf, rk, rkf, w.

[13]Kálmán filter learning produces faster adaptation of expectations, than constant gain learning. It has been shown that constant gain learning, represents as special case of least squares learning, that in turn is a special calibrated version of a hidden factor model (Molavi, 2019). Furthermore, constant gain algorithm and the Kálmán filter have the same asymptotic behavior in large samples, however their transitional dynamics display differences due to learning.

[14]This constant allows to have trending beliefs, and can be thought of as biased expectations about the steady state of the model. This assumption furthermore relaxes the common deterministic growth rate assumptions for the beliefs.

consider PLMs that remain in an eight standard deviation neighborhood of the steady state. Should expectations become unstable[15], I replace them with RE equivalent counterparts.

The Equations 8 and 9 tracks the agents' beliefs over time. Similar to any state space model, the results are very sensitive to the initialization, Slobodyan and Wouters (2012) explore the role of initialization in detail. I select the initial beliefs for both the policy function, Φ, and the state disturbance covariance matrix R, that are compatible with the RE equilibrium. This choice is motivated by the fact that stable, unique constant gain adaptive learning beliefs converge to the RE solution, thus in the steady state RE is an accurate description of what mean beliefs would be.

This also implies that an economy without learning will maintain the RE equilibrium dynamics, and the RE expectations PLMs without responding to central bank announcements. In summary, the baseline SW07 model considered here has 44 state variables. That is y is a 44×1 vector. The beliefs are described by $13 \times (1 + 17)$ by the Φ_t matrix. Where the first dimension is the number of forward-looking states 13, and the second dimension of Z_t: the constant plus MSV variables used to form the beliefs. The mean square error matrix R_t is a $(1+17) \times (1+17)$ squared matrix. Thus for the beliefs the following holds:

$$\underset{18\times 13}{\Phi_{t|t}} = \underset{18\times 13}{\Phi_{t|t-1}} + \underset{1\times 1}{\tau} \underset{18\times 18}{R_{t|t}^{-1}} \underset{18\times 1}{Z_t} (\underset{1\times 13}{y_t^{ALM}} - \underset{1\times 13}{y_t^{f}}) \tag{10}$$

$$\underset{18\times 18}{R_{t|t}} = \underset{18\times 18}{R_{t|t-1}} + \underset{1\times 1}{\tau} (\underset{18\times 1}{Z_t} \underset{1\times 18}{Z_t'} - \underset{18\times 18}{R_{t|t-1}}). \tag{11}$$

3 backward-looking Beliefs in the Smets Wouters Model under Adaptive Learning

In this section, I present the implementation of forward guidance in the baseline Smets Wouters (SW) model when agents have backward-looking expectations. I argue that with backward-looking agents, the extended path simulation (Fair and Taylor, 1983) is the appropriate method to solve for the dynamics of the forward guidance. This method implies that the implementation of the forward guidance path is perceived by agents as a sequence of unanticipated monetary policy shocks, leading to the absence of amplification in the responses as the forward guidance horizon increases.

It is important to note that this result depends on the information structure forming the intial beliefs. I assume that agents do not form expectations based on anticipated news about future monetary policy, as they are not part of the information set. In other words, they only form expectations about shocks that were originally considered in the SW model. The model remains consistent with the baseline SW model by including information beyond the current period about the one-period-ahead price and wage markup shocks.

[15] In the examples considered below, it is never the case.

In contrast, incorporating anticipated news shocks into the model is more appropriate for simulating forward guidance under adaptive learning. While we discuss this approach in detail in Section [Odyssean], it is important to emphasize that anticipated news shocks represent forward-looking beliefs. Starting with backward-looking beliefs is natural, given that the baseline SW model was not designed to include anticipated news shocks about monetary policy. To consider anticipated news shocks, a different model needs to be developed, where agents' information set includes these shocks.

By focusing on backward-looking beliefs in the SW model, we can identify the challenges and limitations of implementing forward guidance. However, to fully capture the effectiveness of forward guidance, it is necessary to incorporate forward-looking beliefs and consider models that explicitly account for anticipated news shocks about monetary policy.

In contrast, incorporating anticipated news shocks into the model is more appropriate for simulating forward guidance under adaptive learning. While we discuss this approach in detail in Section [Odyssean], it is important to emphasize that anticipated news shocks represent forward-looking beliefs. Starting with backward-looking beliefs is natural, given that the baseline SW model was not designed to include anticipated news shocks about monetary policy. To consider anticipated news shocks, a different model needs to be developed, where agents' information set includes these shocks.

3.1 Extended Path Simulation

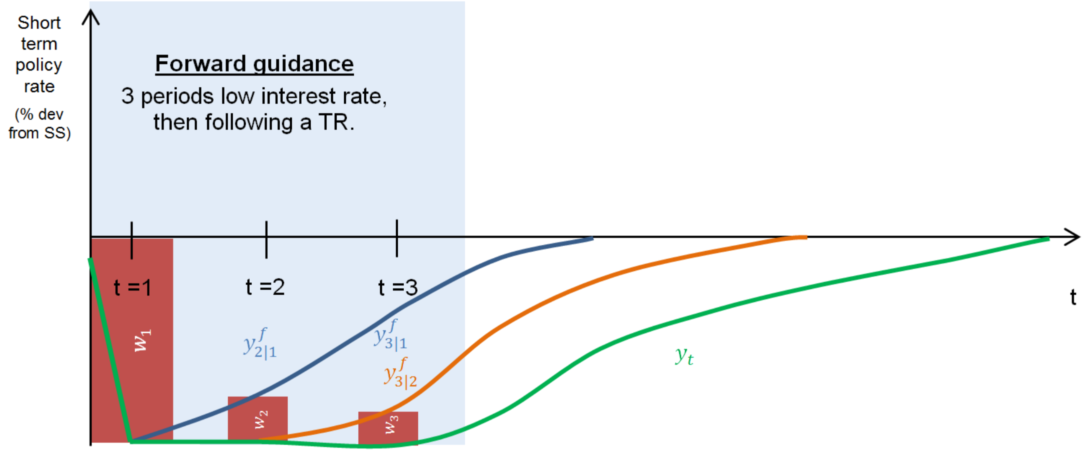
In what follows, I demonstrate how constant gain adaptive learning can overcome the forward guidance puzzle by avoiding amplification of the impact as the forward guidance horizon increases. To simulate the impulse response function (IRF) of forward guidance, I employ the extended path simulation method[16] for the stochastic model.

o conduct the extended path simulation, we need to determine agents' expectations. With backward-looking beliefs only the current and already realized interest rates will matter for the expectations formations. In the case of backward-looking beliefs, only the current and already realized interest rates are relevant for expectation formation, while the announced horizon or path has no impact. Consequently, expectations will not respond to forward guidance. The only effect will stem from the central bank's action in the current period, which will be perceived as an unanticipated monetary policy shock. As a result, the implementation of forward guidance will be perceived by agents as a sequence of unanticipated shocks with diminishing magnitudes.

Consider the scenario where the central bank announces forward guidance in the SW model. Agents' beliefs are initialized at the RE equilibrium and do not incorporate anticipated news shocks, making them inattentive to the forward guidance announcement.

They only learn about the present impact of the forward guidance and perceive the first period of forward guidance as an unanticipated monetary policy shock (w_1 in Figure 1). As

[16] For a quick refresher on extended path simulations see Lawrence Christiano's lecture at Gerzensee in 2014 (Christiano, 2014).



Notes: Figure is only illustrative. It shows how forward guidance under backward-looking beliefs is perceived. The figure shows the PLM dynamics, i.e. the impulse responses of the short-term interest rate. The forward guidance is a time dependent forward guidance. It constitutes of three periods of low interest rate, and a subsequent implementation of the Taylor rule. (Source: Author's calculations)

time progresses, agents follow their policy function and move the economy along the impulse response function (IRF) of the previous period's unanticipated monetary policy shock. This IRF represents agents' expectations if no further shocks occur. These PLMs are shown on Figure 1 with $y^f_{h|t}$, while the actual law of motion (ALM) is represented by y_t. Due to the local stability of the DSGE solution under adaptive learning, the next period's state will deviate from the preannounced path, prompting the central bank to take action. The central bank by delivering on the forward guidance will surprise the agents with a monetary shock that eliminates the difference of the forward guidance path and PLM y, represented w_2.This unanticipated monetary accommodation creates its own IRFs. With the cumulative impact of the previous shock and the monetary accommodation, the PLM becomes more persistent than the RE response to the initial shock. This implies that in the next period the monetary policy accommodation w_3 will have to be even smaller, not only due to the cumulative impact of the past shocks, but due to the adaptive learning as well. Formally this means that conditional on first period information expectations forecasted h periods ahead will be the IRF of the RE solution to an unanticipated shock: $y^f_{h|1,w_t,\tau} = y^f_{h|w_t,RE,\tau}$, where the shock's contemporaneous impact is $R_{1|1,\tau}$. Note the dependence of the beliefs on τ, it highlights that the larger the constant gain parameter in the learning the more more persistence it will create. The change in the persistence of the dynamics, represented by the policy function, is driven by the expectation error that agents make compared to the forward guidance. In the first period, agents experience the expectation error to the full extent of the monetary policy shock. However, as time progresses, agents learn the new path and adjust their expectations even more, with the extent of adjustment determined by their gain parameter.

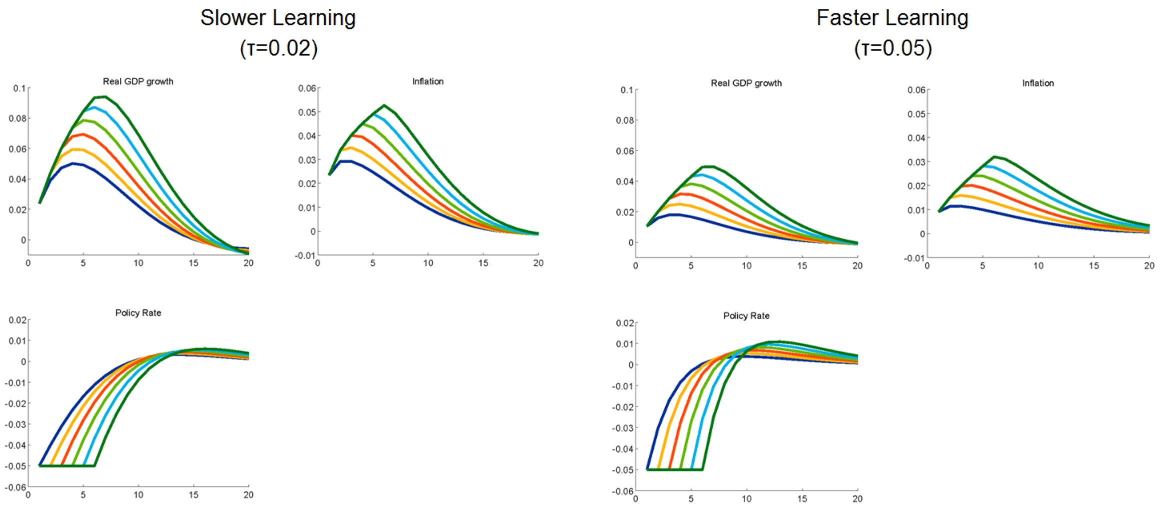
This diminishing impact of the shock is the reason why forward guidance puzzle is resolved.

The marginal impact of an additional horizon of forward guidance is zero in the period of the announcement and becomes progressively smaller in the period of implementation.

3.2 Simulation of Forward Guidance with backward-looking Beliefs

Figure 2 s displays the impulse response functions (IRFs) of selected variables to forward guidance in the SW model. The forward guidance involves setting the interest rate to -0.05 percent per quarter for horizons ranging from 1 to 6 periods. The different colors represent different horizons.

Figure 2: Forward Guidance in the SW07 Model with backward-looking Beliefs



Notes: Forward guidance of setting the interest rate at -0.05 (quarterly rate) for 1-6 horizon, then following the model's Taylor rule. The blue color represents one horizon, yellow two, red three, green four, light blue five and dark green six periods of low interest rates. The model was solved using the AL tools of the MMB. The model is initiated in the RE SS as in Slobodyan/Wouters (2012). A larger τ means more adaptive expectations and more learning "away" from the RE dynamics. Similarly a lower τ translates to slower learning, less adaptive expectation, agents stick more their RE dynamic beliefs. (Source: Author's calculations)

The IRFs are computed using extended path simulation and represent the actual law of motion (ALM) under different degrees of learning[17]. A higher value of τ corresponds to more adaptive expectations and faster learning, while a lower value of τ indicates slower learning and greater adherence to rational expectations (RE) dynamics. The figure illustrates that the forward guidance puzzle is resolved present.

[17] Evans and Honkapohja (2012) show that setting τ close to ∞ the beliefs are those of RE equilibrium. Note as the model is initialized at the RE equivalent setting τ to zero, i.e. switching off learning, will also deliver RE solution.

4 forward-looking Beliefs under Adaptive Learning

In this section, I discuss the modeling approach for forward-looking adaptive beliefs in the presence of a credibly committed forward guidance.

To describe a credible commitment, Andrade et al. (2019) draw an analogy to Odysseus' journey and the Island of the Sirens. Odysseus had his sailors tie him firmly to the ship's mast and put beeswax in their ears to resist the seductive song of the sirens, literally making a "binding commitment" that allowed the ship to safely pass the Island. Similarly, the central bank can make a credible commitment to stick to a preset forward guidance path and resist deviating from it for short-term benefits.

The concept of an Odyssean commitment suggests that the central bank can achieve the optimal outcome by making a credible case for forward guidance. However, as Gertler (2017) demonstrate, if initial beliefs already understand and incorporate the forward guidance commitment, the forward guidance puzzle will re-emerge. In their study of credible forward guidance in a hybrid expectations model, agents had to learn about the projected deviations from the Taylor rule due to forward guidance. In contrast, I argue that if expectations already account for credibly anticipated news shocks, agents will only need to learn about the variance of these shocks as they realize, while the steady state of the economy remains unchanged.

Anticipated news shocks provide a framework to study agents' response to information about anticipated events or shocks. They have been widely used in the DSGE literature and are implemented as lagged structural shocks capturing the information content of future policy actions. Incorporating anticipated news shocks requires expanding the baseline model's Taylor rule to include the moving average representation of these shocks. This change in the model's structure is necessary to capture the impact of anticipated news shocks and their role in shaping agents' expectations.

Thus incorporating anticipated news shocks in the SW07 model requires modifying the Taylor rule and expanding the model's structure. This extension allows for the analysis of forward-looking adaptive beliefs in response to a credibly committed forward guidance.

$$r_t = \rho r_{t-1} + (1-\rho)(\theta_\pi \pi_t + \theta_x x_t) + \varepsilon_t^R + \sum_{l=1}^{L} \varepsilon_{t-l}^{R,FG,l} \tag{12}$$

Anticipated news shocks in this representation are announcements by the central bank in period $t-l$ that the interest rate will change l periods later, i.e. in period t. They can be implemented in a DSGE with the help of new auxiliary variables $f_{k,t}$, that capture the cumulative impact of the forward guidance for periods beyond k and have a recursive definition, as proposed by Laséen and Svensson (2011):

$$f_{k,t} = f_{k+1,t-1} + \varepsilon_t^{R,FG,k}. \tag{13}$$

If the central bank communicates guidance on the interest rate for L periods ahead, there would be $1,2,3...L$ forward guidance shocks that affect the monetary policy rule in period t, and thus the Taylor Rule can be written recursively as follows:

$$r_t = \rho r_{t-1} + (1-\rho)(\theta_\pi \pi_t + \theta_x x_t) + \varepsilon_t^R + f_t, \tag{14}$$

$$f_t = f_{1,t-1}, \tag{15}$$

$$f_{1,t} = f_{2,t-1} + \varepsilon_t^{R,FG,1}, \tag{16}$$

$$f_{2,t} = f_{3,t-1} + \varepsilon_t^{R,FG,2}, \tag{17}$$

$$f_{3,t} = f_{4,t-1} + \varepsilon_t^{R,FG,3}, \tag{18}$$

$$\vdots \tag{19}$$

$$f_{L,t} = \varepsilon_t^{R,FG,L}, \tag{20}$$

Note the missing index for the auxiliary variable f_t as it is the cumulative forward guidance shock in period t entering the Taylor rule, f_t is the sum of all news shocks revealed in period t: $f_t = \sum_{l=1}^{L} \varepsilon_{t-l}^{R,FG,l}$.

4.1 Smets Wouters Model with forward-looking Beliefs

forward-looking beliefs in the SW07 model require additional exogenous state variables.

The model variables therefore will be described by:

$$\{y_{SW07+L}\} = \{y_{SW07}\} \cup \{f_t\} = \{y_{SW07}\} \cup \{f_{1,t}, ...f_{L,t}\} \tag{21}$$

With forward-looking beliefs, all information about the forward guidance is known in period t. Initializing beliefs at the RE equilibrium with news shocks leads to learning only on impact, as the economy follows agents' perceived law of motion (PLM) beyond the first shock.

Thus the belief matrix in terms of dimension has the following format:

$$\underset{(18+L)\times 13}{\Phi_{t|t}} = \underset{(18+L)\times 13}{\Phi_{t|t-1}} + \underset{1\times 1}{\tau} \underset{(18+L)\times(18+L)}{R_{t|t}^{-1}} \underset{(18+L)\times 1}{Z_t} (\underset{1\times 13}{y_t^{ALM}} - \underset{1\times 13}{y_t^f}) \tag{22}$$

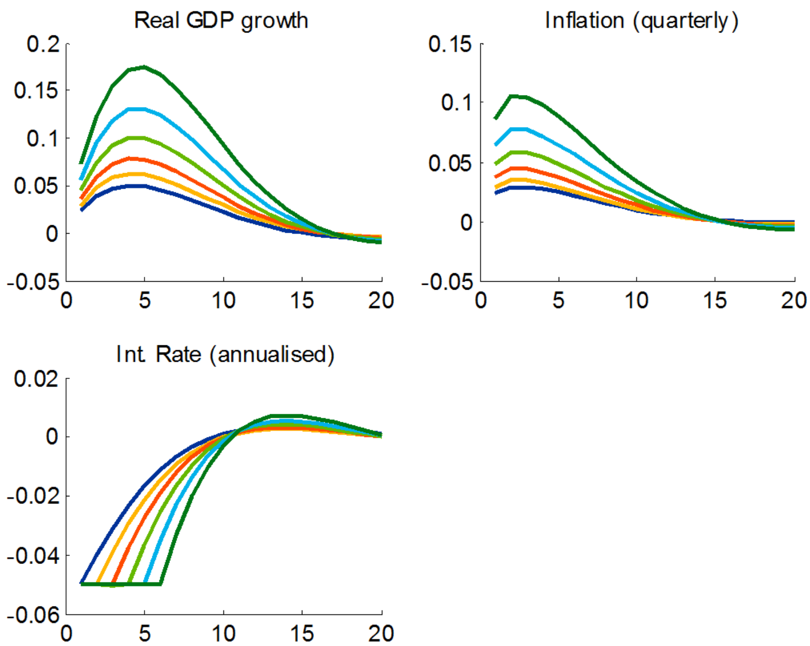
$$\underset{(18+L)\times(18+L)}{R_{t|t}} = \underset{(18+L)\times(18+L)}{R_{t|t-1}} + \underset{1\times 1}{\tau} (\underset{(18+L)\times 1}{Z_t} \underset{1\times(18+L)}{Z_t'} - \underset{(18+L)\times(18+L)}{R_{t|t-1}}). \tag{23}$$

Figure 3 shows the IRFs of selected variables to a forward guidance of setting the interest rate to -0.05 percent per quarter for 1-6 horizon. As before blue color represents one horizon, yellow two, red three, green four, light blue five and dark green six periods of low interest rates. The IRFs are plotting the ALM that coincides with the PLM.

To account for adaptive learning, the model has to be solved forward based on the current period beliefs. Depending on the size of the shock the agents will learn not only the actual path implemented but update the long-run state covariance matrix, the mean squared error matrix R_t. Based on the updated beliefs Φ_t the agents will make expectations about the collection of shocks that implements the forward guidance path. The idea for the solution to be forward-looking is to ensure that the anticipated path of the short-term interest rate, described by the PLM, has to coincide every period with that announced by the central bank. To solve the model forward, I developed a global solution method by iterating on the conditional forecasts in order to find the period t perceived FG shocks: both current period monetary $\varepsilon^{R}_{t|t}$ and the sequence of anticipated news shocks $\sum_{l=1}^{L} \varepsilon^{R,FG,l}_{t-l}$ that implement the forward guidance path. Formally shocks are computed to match the mean expected interest rate path to the forward guidance. This non-unique mapping ensures that under period t expectations the monetary policy shock and anticipated news shock implement a forward guidance of L periods at the path announced by the central bank.

This implies that every period the forward-looking agents re-anchor their expectations to the path, anticipating a new sequence of shocks [18].

Figure 3: SW07 Model with forward-looking Beliefs



Notes: Forward guidance of setting the interest rate at -0.05 (quarterly rate) for 1-6 horizon, then following the model's Taylor rule. The blue color represents one horizon, yellow two, red three, green four, light blue five and dark green six periods of low interest rates. The model was solved using the AL tools of the MMB. The model is initiated in the RE steady state as in Slobodyan/Wouters (2012). (Source: Author's calculations)

Figure 3 illustrates that even with adaptive learning forward-looking belief with news shocks is still subject to the forward guidance puzzle. Learning takes place over time, with the largest

[18] In contrast to Cole (2015), who only solved for the initial period shock and did not ensure that PLMs also match the path every period, I need to ensure that beliefs are consistent with the FG, every period, as this feature will be relevant for the regime-switching belief case.

forecast error made by the agents upon announcement.

5 Endogenous Beliefs Switching

The concept of endogenous belief switching combines endogenous regime switching DSGE with adaptive learning. The idea of endogenous belief switching relies on two key elements. First, beliefs need to be time-varying, which is achieved through adaptive learning. Second, the switching of beliefs should be based on the likelihood of current information.

An analogy to understand endogenous belief switching is to view agents as very sophisticated econometricians. They try to identify the best forecast model by combining the two regimes' solutions: the backward-looking and forward-looking AL versions of the DSGE. That is, agents are aware of the two regimes. They form optimal expectations based on the regimes and combine their two forecasting models using the regime probability weights, after having observed the forecast errors, they update their regime-dependent beliefs and then the regime probabilities.

To best understand endogenous belief switching, consider a simple example. If there is no historical precedent of forward guidance and anticipated news shocks, agents would learn to exclude the yield curve from their econometric model, as it would have no significant explanatory power beyond the mean squared variance and the true dynamics of the model. In this case, agents would converge on a coefficient of zero for the yield curve regressors in their forecast model.

However, if there is a history of credible forward guidance commitments, agents would learn the true dynamics of the model, where past monetary policy shocks also matter. This would result in VARIMA models with integration in the moving average terms. The stability of such models is the foundation of the forward guidance puzzle. Therefore, it is reasonable to expect that econometricians would reject a model that exhibits local instability in the economy. If the forecast model for forward-looking beliefs generates impossible predictions, agents would switch to a stable VAR model, that is backward-looking. In previous sections, I have established how forward guidance works if agents do or do not form beliefs about anticipated shocks. I have argued that the baseline DSGE models, like the SW07 model, (usually) do not feature anticipated news shocks, and thus standard models with constant gain adaptive learning beliefs will not feature the forward guidance puzzle. Furthermore, even though backward-looking agents will perceive the forward guidance as a sequence of unanticipated monetary policy shock, due to learning they will respond more and more persistently to it, as it gets implemented.

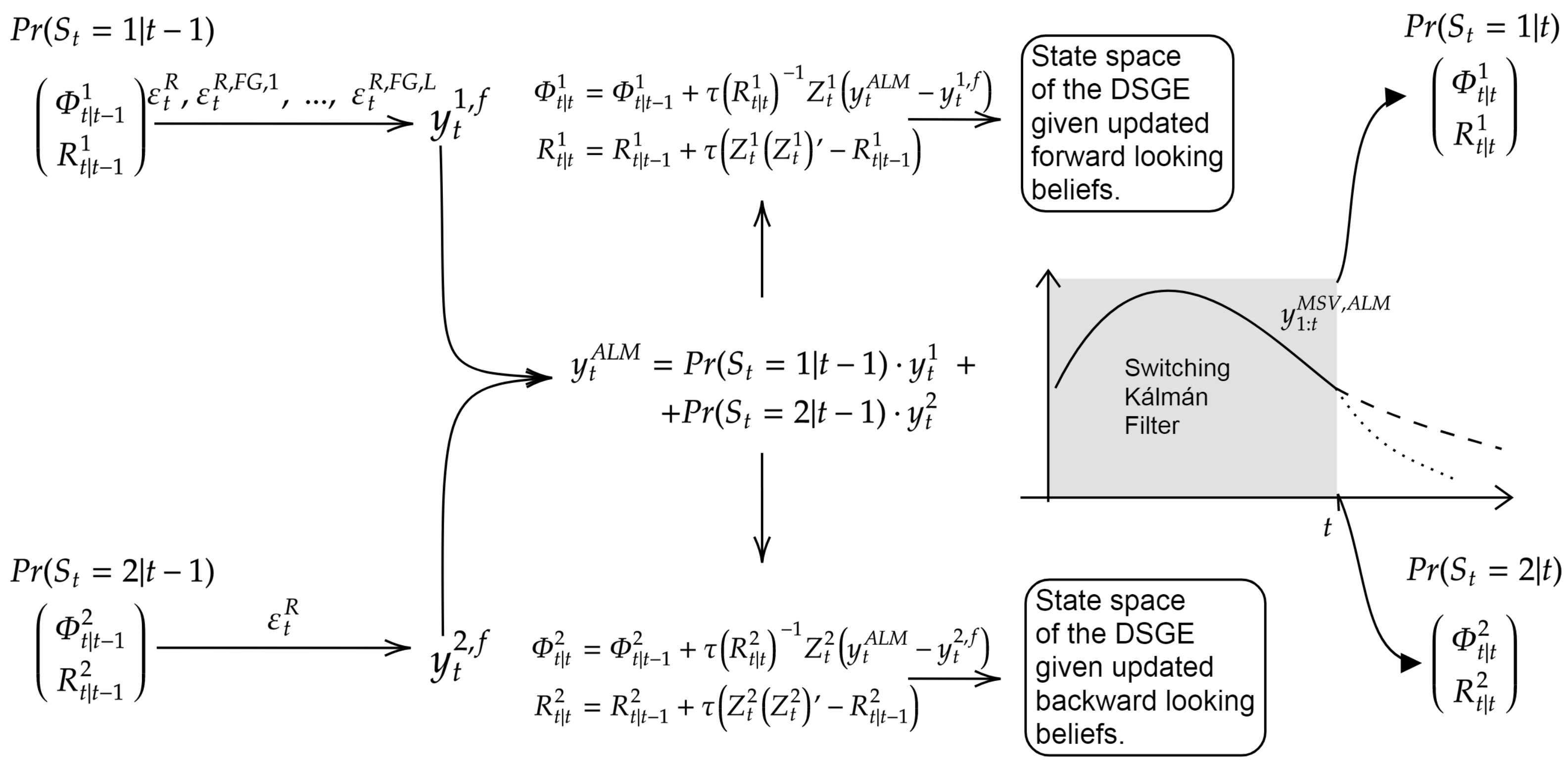
In what follows I present the ingredients of endogenous belief switching: First, beliefs should be time-varying. This is ensured by adaptive learning. Second, switching of beliefs should take place based on the model fit, i.e. likelihood estimated using a switching Kálmán Filter, leading to an endogenous regime switching.

Endogenous regime switching has been proposed to multiple models when agents have rational expectations. Lansing (2018) used endogenous regime switching at the zero lower bound. By allowing for expectations to feature regime switching between two local equilibria, labeled the ”targeted” and ”deflationary” regimes, he studied the role of central bank anchoring. In his model the expectations were a result of model averaging, and switching was driven by the root mean squared forecast error of beliefs given a logistic probability. Similar to his model, endogenous belief switching is a mixed expectations equilibrium, where mixing is not following a logistic rule, but using a Kálmán filter to estimate the perceived likelihood of the belief's state representation factor model. As in Lansing (2018); Bullard and Duffy (2004) I assume agents are aware of the two local beliefs, and the resulting self-fulfilled equilibria of the forward and backward-looking beliefs[19].

Endogenous belief switching is summarized on Figure 4. It shows that initial beliefs, on the left, are updated through learning, in the middle, giving rise to state representation of the regimes and that allow for belief switching, on the right. To understand the role of learning and of belief switching let us walk through the diagram.

[19] In contrast to Bullard (2010) there is no concern about the possibility of being stuck in liquidity, deflationary trap, as the economies implied by the beliefs feature the same asymptotic steady states

Figure 4: Endogenous Belief Switching



(Source: Author's illustration)

5.1 Timing Assumptions and Adaptive Learning

Agents in the model start each period with an a priori belief about the probability of each regime, $Pr(S_t = i|t-1)$. They inherit this knowledge from the previous period in the form of belief matrices. They observe the relevant shocks based on the information set of the regime they are considering. For forward-looking beliefs, they observe the current monetary policy shock and the sequence of anticipated news shocks implementing the forward guidance, $\varepsilon_t^R, \varepsilon_t^{R,FG,1}, ..., \varepsilon_t^{R,FG,L}$. For backward-looking beliefs, she only observes the current monetary policy shock ε_t^R.

Agents then form their PLMs, $y_t^{i,f}$, based on their beliefs and the observed shocks. It is important to note that agents must respond first and then learn, as this timing restriction avoids simultaneity problems.

Using the a priori probabilities, agents weigh the regimes and act accordingly. This subjective response is the best agents can make given the available information and their learning assumptions. This leads to the creation of the ALM which combines the two PLMs based on their respective probabilities:

$$y_t^{ALM} = Pr(S_t = 1|t-1) \cdot y_t^1 + Pr(S_t = 2|t-1) \cdot y_t^2 \tag{24}$$

Agents are then allowed to update their beliefs using the updating equations. Learning takes place as long as the PLMs for the two beliefs are different, resulting in expectation errors and subsequent updating. However, if only one regime is active, it will determine the ALM and will only result in minor forecast errors, eventually leading to the convergence of regimes to a shared steady-state that can be used to solve the models.

Then the updated beliefs consistent DSGE model is transformed into a state space representation. These state representations of the beliefs are used to update the a priori belief by evaluating which regime fits the observed MSV state of the IRF better. When computing regime fits,i.e. likelihoods, agents filter the observed MSV states to adapt their beliefs and reinterpret past states in light of their current beliefs. Finally, agents switch beliefs by evaluating the likelihood of perceived shocks in relation to the observed states of the MSV. This is done using the switching Kálmán Filter (Murphy, 1998).

I assume that the agents start out every period with an a priori belief about the probability they assigns to the two beliefs $Pr(S_t = i|t-1)$. The agents inherit this knowledge from the previous period in form of belief matrices under both regimes $\Phi^i_{t|t-1}$ and $R^i_{t|t-1}$. The agent then observe the stochastic shock(s) depending on the information set of the regime. For forward-looking beliefs, they observe the current monetary policy shock and the sequence of anticipated news shocks implementing the forward guidance, $\varepsilon_t^R, \varepsilon_t^{R,FG,1}, ..., \varepsilon_t^{R,FG,L}$. For backward-looking beliefs, she only observes the current monetary policy shock ε_t^R. Given beliefs and the shocks, the agents form the PLM for both beliefs: $y_t^{i,f}$. Note that no observation of the states is necessary until this stage. The ALM and both of the PLMs are known by the agents. Subsequently, the agents are allowed to update both beliefs using 8 and 9. Here is where learning is taking place: as long as PLMs of the two beliefs are different expectation errors will arise and lead to updating. However if one of the regimes is active only the other will get updated. This will lead to an eventual convergence of beliefs. This convergence of beliefs is the reason, I build endogenous belief switching on a constant gain adaptive learning framework.

Finally given the updated beliefs the DSGE is cast into the state space representation. These state representations of the beliefs are used to update the a priori belief, which regime fits better the observable MSV state of the IRF:

$$y_t^i = \mathbf{F}(\Phi_{t|t}^i, R_{t|t}^i) y_{t-1}^i + \mathbf{w^i}_t, \tag{25}$$

where y_t^i is the state given belief i, $\mathbf{F}$ is the state-transition matrix, that is dependent on the updated beliefs, and finally the state noise $\mathbf{w^i}_t$ that is assumed to be normally distributed with mean zero and covariance matrix $Q_t^i = R_{t|t}^i \Sigma \left(R_{t|t}^i\right)'$. Where Σ is a diagonal matrix with the size of the exogenous shocks, providing the scaling of their variances. Therefore the state noise is multivariate normal: $\mathbf{w^i}_t \sim N\left(0, Q_t^i\right)$.

Using the a priori probabilities the agents weight the regimes and act accordingly. Weighting the beliefs with their a priori probabilities is a complex combination of best responses. Therefore it is the best subjective response the agents can make given the information and assumption about learning.

This gives rise to the ALM of the economy:

$$y_t^{ALM} = Pr(S_t = 1|t-1) \cdot y_t^1 + Pr(S_t = 2|t-1) \cdot y_t^2 \tag{26}$$

Note that no observation of the states is necessary until this stage. The ALM and both of the PLMs are known by the agents. Subsequently, the agents are allowed to update both beliefs using 8 and 9. Here is where learning is taking place: as long as PLMs of the two beliefs are different expectation errors will arise and lead to updating. However if one of the regimes is active only the other will get updated. This will lead to an eventual convergence of beliefs. This convergence of beliefs is the reason, I build endogenous belief switching on a constant gain adaptive learning framework.

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The goal of the agents is to make sense of the economy. To that end they will filter the observed MSV states starting from the onset of the monetary policy action agent to adapt their beliefs regarding the shares. This requires them to filter the whole path of the economy, and try to reinterpret past states of the economy in light of the current beliefs. This enables them to reinterpret a consistent delivery of the forward guidance from being a sequence of backward-looking shocks into a credible forward guidance.

The agents switch beliefs by evaluating the likelihood of the perceived shocks' probability in light of the observed states of the MSV using the switching Kálmán filter (Murphy, 1998). For the mathematical details on the switching Kálmán filter please see Appendix 7.

Considerations of the existence and properties of asymptotic equilibrium to which the beliefs converge is beyond the scope of this paper and will be explored in future research[20].

Belief switching enables the acceleration of learning. Recall that backward-looking beliefs adaptive learn the forward guidance implementation as a sequence of contemporaneous monetary policy shock, and become more persistent. However there is relevant surprise to beliefs, and thus expectation errors in the initial period only, if beliefs are forward-looking and get anchored at the path implied by the forward guidance. Endogenous belief switching enables to switch between backward and forward-looking expectations. The switch takes place based on the probability allocated to either of the beliefs. The representative agent chooses the belief to follow that has the highest (posterior) probability to have generated the observed path. This probability has a natural interpretation: it captures the confidence of the agents that the central bank forward guidance commitment will be followed, thus it is a measure of central bank credibility.

5.2 Belief Switching Illustration in One Dimension

To illustrate how adaptive learning with endogenous belief switching works consider a simple example. First, we need to answer what is the minimal number of states that is required to make switching possible. If the MSV is univariate, that is, the only state variable is enough to characterize the economy, the only state we can have should capture monetary policy. Denote this state as the short-term rate. The problem of only one dimension is that agents have no way to tell during forward guidance the two regimes apart since the only observable is the short-term rate. Recall, that the central bank implements forward guidance by setting the short-term interest rate to a pre-announced path for a given number of periods. Since this state variable follows the same path under both beliefs, there is only room to switch beliefs on the

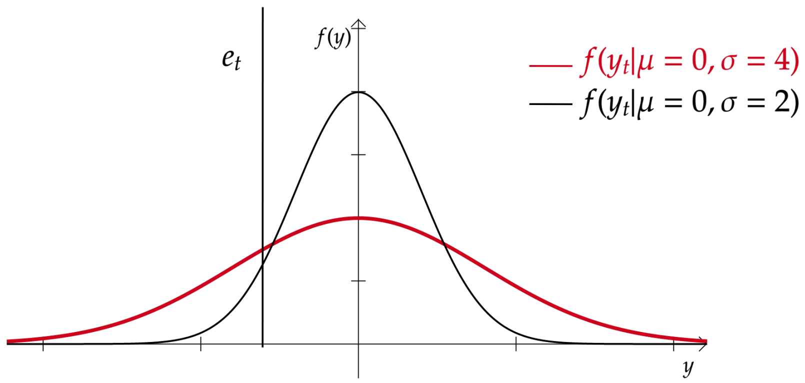
[20] Simulations indicate that without shocks the backward-looking beliefs, due to featuring less uncertainty are the absorbing regime. However depending on the assumption of the signal to noise ratio, alternative equilibrium can arise: e.g. if the observation error and thus noise around the model are large, there will be no distinction between the beliefs in terms of probability and no endogenous switching will take place. Then the asymptotic equilibrium will be centered around the equilibrium where adaptive beliefs make no errors given an ALM of 50% of each beliefs.

announcement. As every period is the same irrespective of the beliefs. The short-term rate has the same path under both regimes.

For the purpose of simplicity, assume that a priori beliefs are backward-looking. Then all it suffices is to calculate the likelihood function of the states given switching takes place. Assume then that the model with forward-looking beliefs is filtered as a normal with mean zero and 4 standard deviation denoted as $f(y_t|\mu = 0, \sigma = 4)$. Similarly, the agent has knowledge about the state distribution, but beliefs remain backward-looking. backward-looking beliefs feature fewer structural shocks, and fewer sources of uncertainty and thus display a smaller variance than forward-looking beliefs. Assume, that the backward-looking beliefs regime has the same mean and half the variance, e.g. 2 standard deviation, $f(y_t|\mu = 0, \sigma = 2)$.

Figure 5 displays the state densities given initial backward beliefs: red is the sate distribution if switching to forward-looking beliefs takes place. It displays a more dispersed short-term rate. The black line is the state distribution with backward-looking beliefs. Assume the central bank

Figure 5: Belief Switching Illustration - Initial Beliefs

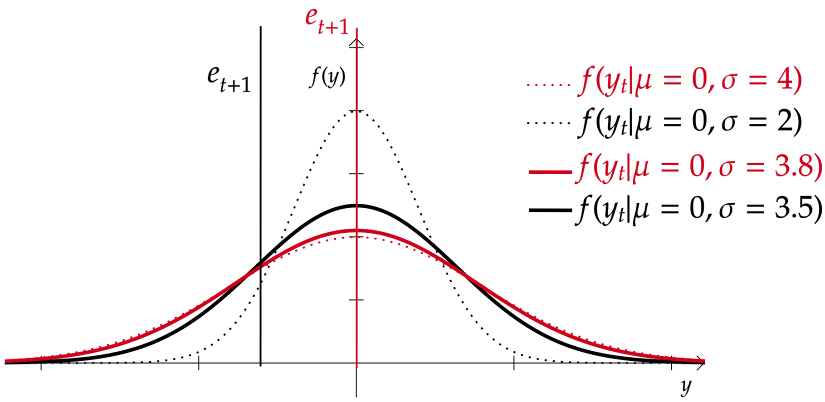


(Source: Author's illustration)

announces forward guidance for two periods. Since the model's observable dimension is one, and no monetary policy action was expected, the current shock is perceived almost as the same movement in the short-term rate, setting it to the path.

Evaluating the likelihood in this case is simple, it means finding the model that attributed higher probability to the data. If the size of the perceived monetary policy shock,e_t results in a state of the economy where the probability of switching to forward-looking belief is larger, agents will update their regime probabilities accordingly. Figure 6 illustrates what this happens after the announcement, the dashed lines are the initial beliefs, i.e. the PLMs, and the solid lines are the updated beliefs after announcement.. For the numerical example consider that after learning the resulting standard deviation of the sate is $3,5$ for the backward, and 3.8 for forward-looking model.

Figure 6: Belief Switching Illustration - Second Period Beliefs



(Source: Author's illustration)

With adaptive learning the state variances are updated as well. As Figure 6 shows in the next period the two distributions get closer to each other. forward-looking beliefs are only updated marginally, as the mean was accurate, and the realized shock was smaller than the standard error of the interest rate and thus the variance shrinks. The effect of learning is larger on backward-looking beliefs, as the forecast error was larger. In other words, once beliefs switched to forward-looking, they get updated less, while backward-looking beliefs get updated more.

As now the forward guidance gets implemented, the shock perceived under the beliefs is also change. Under forward-looking beliefs, no more monetary policy shock is present, shown with e_{t+1} in red. That is, as the forward guidance is implemented forward-looking beliefs are confirmed. With backward-looking beliefs the implementation of the path is another current period monetary policy shock, shown in black on Figure 6.

As forward guidance is delivered expectations remain forward-looking.

Therein lies the necessity for the two components of endogenous belief switching: adaptive learning and regime switching, the former enables the evolution of the two regimes over time, slow convergence, the latter enables the faster adjustment of expectations in response to policy.

5.3 The Three Equation New Keynesian Model

Now that we have built intuition for belief switching in one dimension, let us make the next small step towards a general equilibrium framework. Consider the adaptive learning version of the standard three equation New Keynesian textbook (henceforth 3EQ) model following (Ravenna and Walsh, 2006). It is a small model that re-creates the trade-off monetary policy faces between stabilizing the inflation rate versus the output gap. The model in its simplest version consists of the following equations:

1. IS Curve
$$x_t = E_t[x_{t+1}] - \frac{1}{\sigma}(r_t - E_t[\pi_{t+1}]) + \varepsilon_t^{IS} \tag{28}$$
2. Phillips Curve
$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa(\sigma+\eta)x_t + \kappa r_t \tag{29}$$
3. Taylor Rule
$$r_t = \rho r_{t-1} + (1-\rho)(\theta_\pi \pi_t + \theta_x x_t) + \varepsilon_t^R + \sum_{l=1}^{L} \varepsilon_{t-l}^{R,FG,l} \tag{30}$$

Where anticipated monetary shocks only exist if the agent forms beliefs about them. Introducing adaptive learning to this model means replacing the expectations:

$$E_t[y_{t+1}] = E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = y_t^f = \hat{\alpha}_{t-1} + \hat{\beta}_{t-1}' \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \\ r_{t-1} \end{bmatrix} = \hat{\Phi}_{t-1} \cdot Z_t \tag{31}$$

Consider the calibration of the parameters:

Table 1: Parameter calibrations in the 3 Equation model

Parameter	**Value**	**Description**
β	0.99	Discount factor
η	1	Frisch elasticity
κ	0.0858	Slope of the PC
ρ	0.900	Interest rate smoothing
θ_π	0.150	Inflation response
θ_x	0.013	Output gap response
$\sigma_{\varepsilon^{IS}}$	1	Standard error of IS curve shock
σ_{ε^R}	1	Standard error of Monetary Policy shock
$\sigma_{\varepsilon^{R,FG,l}}$	1	Standard error of l period ahead forward guidance shock

Given this parametrization the model's MSV has two dimensions, i.e. there are two predetermined variables.[21] I choose the two states describing the system is the interest rate and

[21] A potential problem that might arise is the non-uniqueness of the state representation of the dynamic system. In macroeconomic models this is overcome by timing assumptions, that clearly separate model variables to states and controls. Mapping macroeconomic RE models to adaptive learning representation requires these assumptions to be clearly defined. I resort to the MSV implied by the timing used by Ravenna and Walsh (2006), and leave the discussion of transfer function representation to future research.

output gap, conveniently mapped by the two structural shocks.

Table 7 shows the impulse response function of the 3EQ model to a forward guidance shock of different horizons. The left panel illustrates the impulse response if beliefs are backward-looking and the agent has a learning coefficient of $\tau = 0.02$. The right panel displays the impulse response of the model variables to the forward guidance shock with forward-looking beliefs and adaptive learning. As the model does not have strong persistence, the forward guidance is relatively short-lived in both cases. The left panel shows the marginal impact of the sequence of unanticipated monetary policy shocks, and thus the marginal effect of implementing the path vanes. This diminishing marginal effect on other variables explains the flattened response of output and inflation during the implementation of the forward guidance path.

Given backward-looking beliefs the long-run covariance matrix of the states is the following:

$$Q^2 = \begin{bmatrix} 1.1462 & 0.6465 \\ 0.6465 & 0.7318 \end{bmatrix}, \tag{32}$$

where the first dimension is the short-term interest rate and the second is the output gap. In terms of scale, the state covariance matrix Q is small. The scale can be thought of as the measure of the state dispersion, best characterised by Q matrix's mean singular value, i.e. trace per dimension being 0.9390, that is smaller than 1.

The right panel displays the forward guidance puzzle. Notice the increasing marginal effect on impact for both output and inflation, as the length increases so does the impact of forward guidance.

The long-run covariance matrix of the forward-looking beliefs is the following:

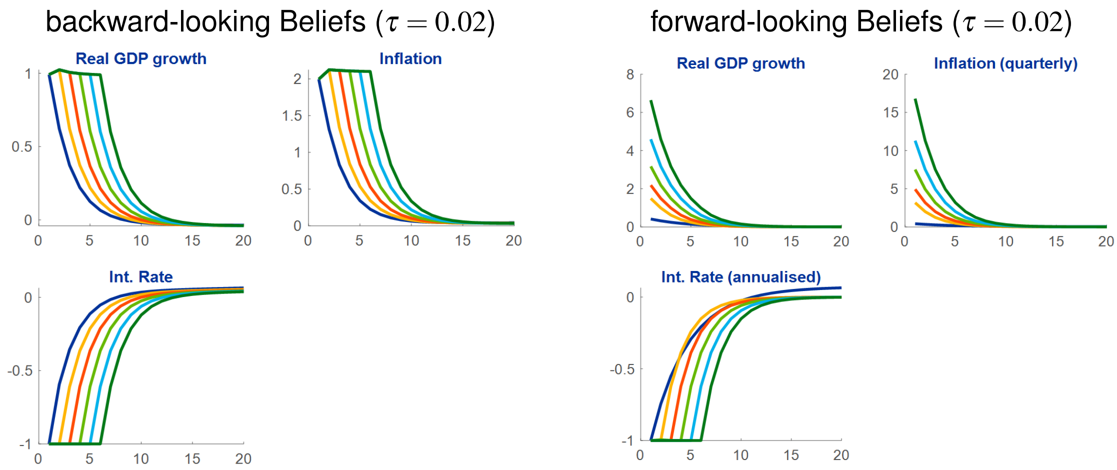
$$Q^1 = \begin{bmatrix} 2.1272 & 0.9927 \\ 0.9927 & 0.8580 \end{bmatrix} \tag{33}$$

As expected, the interest rate has a larger variance than before, while the correlation between the two states is also increased. The scale of the state covariance matrix in terms of its mean singular values is also higher 1.4926. This tells, that forward-looking beliefs generate a state space with higher dispersion.

Note the different dynamics of the unanticipated monetary policy shock seen as the different IRFs in blue between the two panels. It has a smaller impact under forward-looking beliefs. The source of the difference is due to adaptive learning. As the forward-looking model has a larger variance, agents learn less from the forecast error of a given size compared to being backward-looking, where the same size of forecast error makes them update their beliefs more.

This is further supported by the observation, that learning accelerates as the amplitude of the impulse response increases. A higher amplitude results in larger forecast errors, and as learning is constant gains weights on least squares updating, a higher forecast error given a variance translate into accelerated learning.

Figure 7: Forward Guidance Shock in 3EQ Model with Adaptive Learning



Notes: Forward guidance of setting the interest rate at -1 for 1-6 horizon, then following the model's Taylor rule. The blue color represents one horizon, yellow two, red three, green four, light blue five and dark green six periods of low interest rates. The model was solved using the AL tools of the MMB. The model is initiated in the RE solution. For further reference see: Wieland et al. (2012)
(Source: Author's calculations)

Having seen the two extremes, let us turn to endogenous belief switching.

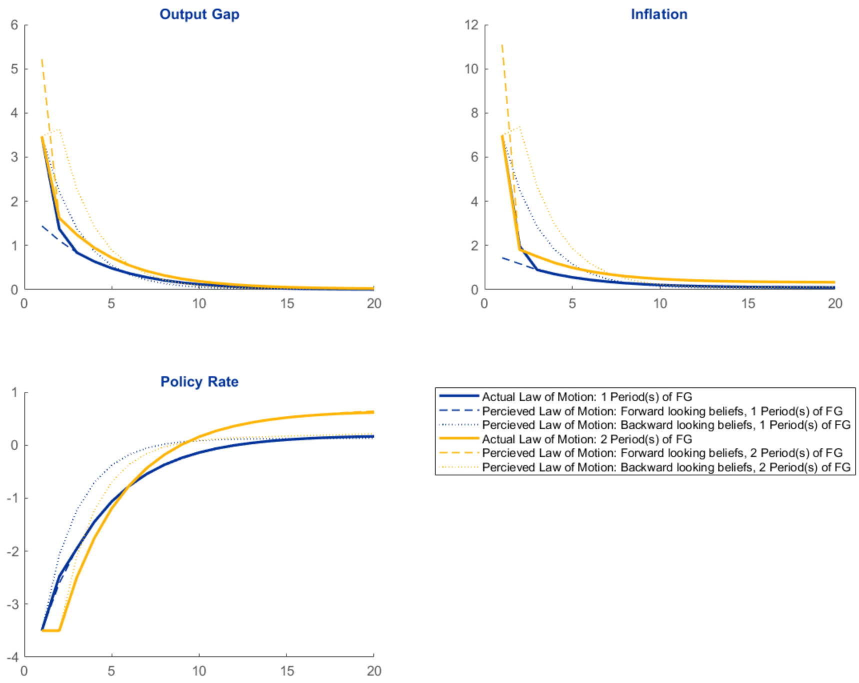
5.4 Endogenous Belief Switching in Higher Dimensions

The following simple illustration is meant to show, how backward-looking agents can switch to be forward-looking. It highlights the role of belief switching in response to the central bank's forward guidance.

Consider an economy, where the a priori share of forward-looking agents is 0. In this backward-looking environment, the central bank announces the forward guidance of 2 periods setting interest rate lower than the equilibrium, by 3.5%, i.e. three and a half times the standard deviation of the average monetary policy shock for the next two periods. It is important for the sake of illustration that the size of the shock is chosen to be very large in comparison the average monetary policy shock. This will enable switching as it will be, by design, very unlikely that such a huge response could have been caused by a single monetary policy shock. In contrast, if beliefs are forward-looking then the variability of the states is larger, thus the size of this shock will be more probable and command a larger likelihood.

Figure 8 shows the IRFs of all three variables to a forward guidance of setting the interest rate to -3.5 percent for 1 and 2 horizons.

Figure 8: 3EQ Model - Endogenous Belief Switching



Notes: The Figure shows the IRF of the state variables in the three equation New Keynesian model to a forward guidance shock when initial beliefs are backward-looking. Forward guidance shock means setting the interest rate at -3.5 (quarterly rate) for 1 and 2 horizons, then following the model's Taylor rule. The blue color represents one horizon, yellow two horizons of low interest rates. The dotted line represents the perceived law of motion for the current period under backward-looking beliefs. The dashed lines show the perceived law of motion of variables under forward-looking beliefs. (Source: Author's calculations)

As before blue color represents one horizon, yellow two horizon forward guidance of low interest rates. The solid lines show the ALM of the model, that is the weighted average of the PLM under each regime. backward-looking beliefs' PLM is also displayed in dotted lines, while the forward-looking beliefs' PLM is shown with a dashed line. Recall, initial beliefs are backward-looking: since beliefs switch only after having responded to the shock, the ALM of the initial impact is overlapping with the PLM of the backward beliefs. In period 0, when the central bank makes the announcement both beliefs will generate a PLM. Since backward-looking beliefs are only surprised by the realization of the shock, not only forward, but backward-looking beliefs will make forecast errors, agents learn and beliefs get updated. In other words, neither regime would have anticipated such large monetary policy accommodation. Given initial, RE beliefs the agents form the PLMs. The PLM of backward beliefs will be the ALM. In response to this forward-looking beliefs change from the RE to a larger variance:

$$Q^1_{RE} = \begin{bmatrix} 2.1272 & 0.9927 \\ 0.9927 & 0.8580 \end{bmatrix} \rightarrow Q^1_{1,AL} = \begin{bmatrix} 2.7303 & 0.7438 \\ 0.7438 & 0.9607 \end{bmatrix} \tag{34}$$

Similarly backward-looking beliefs will learn the larger short-term rate variance, as on average only a size of 1 would be compatible with the RE beliefs, while a zero shock in the IS curve is also not anticipated:

$$Q^2_{RE} = \begin{bmatrix} 1.1462 & 0.6465 \\ 0.6465 & 0.7318 \end{bmatrix} \rightarrow Q^2_{1,AL} = \begin{bmatrix} 1.3425 & 0.4519 \\ 0.4519 & 0.9247 \end{bmatrix} \tag{35}$$

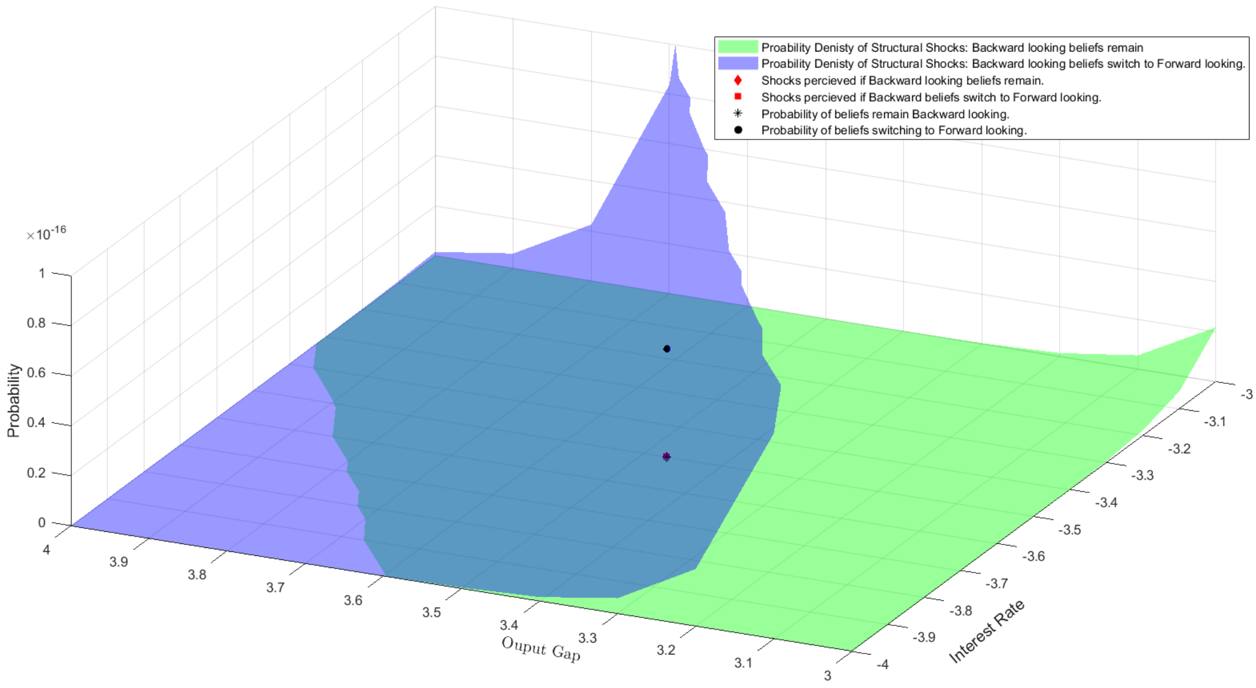
Once the ALM is realized, and learning took place, the agents will update their a priori beliefs about the probability of the regimes, i.e. the probability attached to credible forward guidance. In doing so they will back out the probability that the response could have been a result of backward and forward-lookingbehaviorss, respectively. Finding that such a large movement in the states could have been only generated by an economy where the central bank can credibly commit to forward guidance, the agents choose to switch to forward-looking. Thus the next period the PLM of forward-looking will be the ALM of the economy. Figure 8 shows this as the dashed lines, forward PLM, becomes overlapping with the solid line, ALM, revealing the dotted line, the backward-looking beliefs. Since no further shocks hit the economy,i.e. as long as the forward guidance is not deviated from, forward-looking beliefs prevail.

To understand the switching in detail, let us reconstruct the figures representing the state probabilities under both regimes. Figure 9 shows the initial, updated belief distribution of the MSV under both regimes in the first period [22]. It maps any combination of the state space to a probability. The green surface is that of the initial beliefs remaining backward-looking, while the blue surface represents the probability of the states given backward beliefs switching to forward-looking. Note that both have zero mean, i.e. the steady states are shared. Furthermore forward-looking beliefs result in larger current interest rate state variation compared to backward-looking beliefs in the MSV. Therefore the blue multivariate normal state distribution is more spread out. Recall that the agents start out with backward-looking beliefs, and thus the economy follows the PLM of the backward-looking beliefs upon a shock resulting in a drop -3.5 of the interest rate and an increase of output gap to 3.47. This is the ALM of the model. Note that no prior anticipated shocks were announced, thus any movement in the states is seen as a shock. Therefore the red rhombus, showing the perceived shocks if beliefs remain and the red square showing the perceived shocks if beliefs switch are overlapping. However the likelihood that shocks originated from either of the regimes is different. The likelihood that backward-looking beliefs could have generated this shock is $7.3279 * 10^{-23}$ [23] in other words, it is perceived as almost impossible that backward-looking beliefs could have generated such a large shock.

[22] For a birds eye picture on the state covariance please see Figure 12 in Appendix 7.

[23] To compute the likelihood that backward-looking beliefs prevail the agents compute $L^{2(2)} = \sqrt{det(Q^2_{1,AL})} \cdot exp^{-\frac{1}{2}\Sigma e_1^{2(2)} Q^2_{1,AL} e_1^{2(2)}}$, where $e_1^{2(2)} = [-3.53.47]$ is the perceived shock in period 1 if beliefs remain backward-looking, and $Q^2_{1,AL}$ is the updated covariance matrix of the states in period 1 given backward-looking beliefs.

Figure 9: Belief Switching Illustration - Backward to Forward Beliefs in the 1st Period



(Source: Author's illustration)

The likelihood that forward-looking beliefs generated the economy is $4.3484 * 10^{-17}$ [24], magnitudes larger. This results in a log-likelihood ratio of the two beliefs 26.78 in support of the forward-looking beliefs. It is very strong evidence in favor of forward-looking model being a better characterization of the actual economy. As a result, backward-looking beliefs switch to forward-looking.

At the beginning of the second period adaptive learning takes place. This time the economy follows the path of PLM of forward-looking beliefs, characterized by a fast decay. Thus the covariance matrix shrinks for both beliefs: neither the IS equation shock nor additional forward guidance shocks materialize. Both beliefs get updated, forward-looking beliefs learn to have a more correlated state distribution, and smaller variance, than before:

$$Q^1_{1,AL} = \begin{bmatrix} 2.7303 & 0.7438 \\ 0.7438 & 0.9607 \end{bmatrix} \rightarrow Q^1_{2,AL} = \begin{bmatrix} 2.1174 & 0.9993 \\ 0.9993 & 0.8663 \end{bmatrix} \tag{36}$$

[24]The likelihood that backward-looking beliefs switch to forward-looking is given by $L^{1(2)} = \sqrt{det(Q^1_{1,AL})} \cdot exp^{-\frac{1}{2}\Sigma e_1^{1(2)} Q^1_{1,AL} e_1^{1(2)}}$, following the notation in the Appendix on Switching Kálmán filter, where $e_1^{1(2)} = [-3.53.47]$ is the perceived shock in period 1 if beliefs switch forward-looking, and $Q^2_{1,AL}$ is the updated covariance matrix of the states in period 1 if beliefs switch forward-looking.

The DSGE under forward-looking beliefs has one additional auxiliary state to capture the anticipated shocks, it is however not observable, thus the DSGE estimation involves an emission matrix H^1 that eliminates this auxiliary state. Therefore the likelihood of the model is only measured in the observable state space of only two variables.

Similarly backward-looking beliefs will adapt to the new ALM, making sates more correlated, and variances slightly less dispersed:

$$Q^2_{1,AL} = \begin{bmatrix} 1.3425 & 0.4519 \\ 0.4519 & 0.9247 \end{bmatrix} \rightarrow Q^2_{2,AL} = \begin{bmatrix} 1.0914 & 0.6662 \\ 0.6662 & 0.7616 \end{bmatrix} \tag{37}$$

Therefore in the second period the agent will have forward-looking beliefs, considering the forward guidance fully credible. Now the agents will have to evaluate the probability if it makes sense to switch back to backward-looking beliefs. Figure 10 shows the probability distributions of either remaining forward-looking or switching[25]. As Figure 8 has shown, the ALM in the second period is overlapping with the forward-looking PLM. Therefore, due to being anticipated, forward-looking beliefs have made close to zero error[26]. They see the anticipated implementation of the -3.5 interest rate path and perceive it as almost no error on both the interest rate and output. This is shown on Figure 10 with the red rhombus close to zero.

In contrast when the agent considers backward-looking beliefs, she recognizes, that from that perspective an additional monetary policy shock would be needed to implement the -3.5. In fact, from a backward-looking belief the forward guidance ALM is only achievable when the shock on interest rate is -1.8854, while on output gap it is -0.7997 [27]. Figure 10 illustrates this with the red square. Since the size of the perceived shock is smaller, and thus more probable, under forward-looking beliefs, the agent decides not to switch and remain forward-looking.

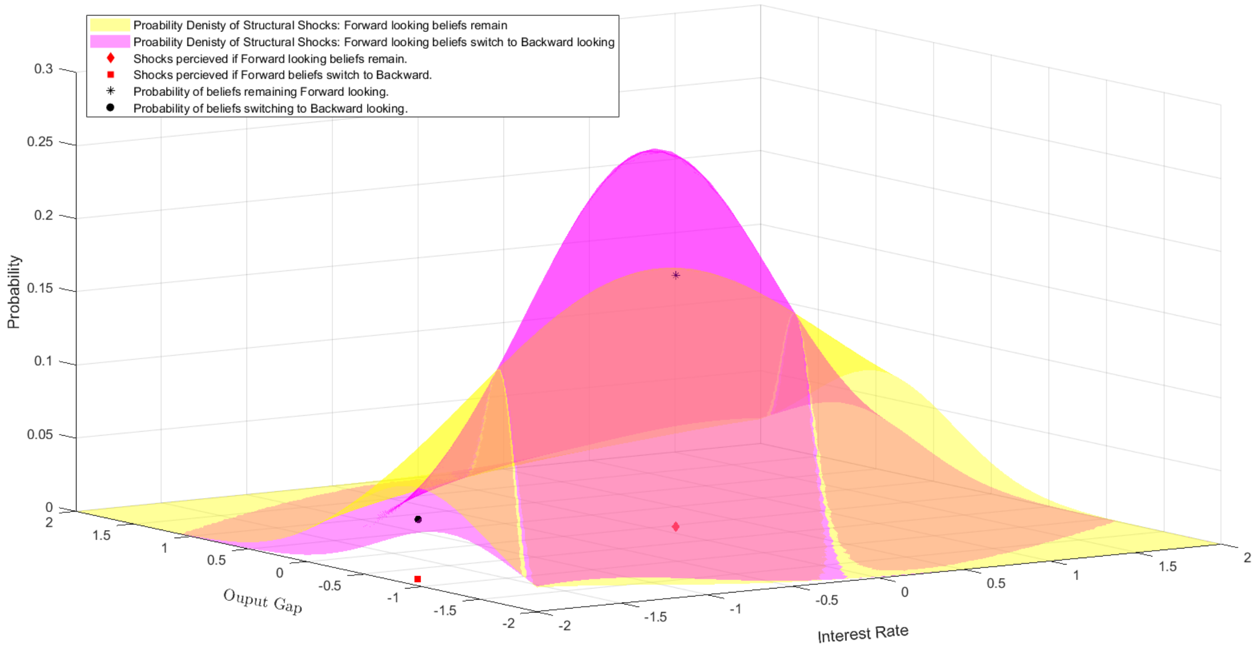
Figure 10 shows this as the black star of remaining lies higher than the black dot of switching.

[25] For another perspective of the same chart please see Figure 13 in Appendix 7.

[26] The errors made are due to the learning of the first period. Since the first period made forward-looking beliefs closer to backward-looking ones, they now perceive a shock that compensates.

[27] This combination of shock is solely do to a monetary policy shock of a size -1.8854.

Figure 10: Belief Switching Illustration - Forward to Backward Beliefs in the 2nd Period



(Source: Author's illustration)

In this simple example, a forward guidance made beliefs switch from backward-looking to forward-looking. The central bank built credibility by the size of the accommodation it made.

It illustrates, that a large size forward guidance will make agent's beliefs forward-looking. However there is another dimension to forward guidance, the length of the spell. A longer forward guidance might be seen as less credible than a shorter one. To illustrate this, the next section will explore how the central bank can determine beliefs, and make backward-looking, inattentive beliefs forward-looking, and thus a forward guidance credible.

5.5 Central Bank and Delivering Forward Guidance: Gaining Credibility

Endogenous belief switching enables to study the credibility of forward guidance from a new perspective. With endogenous belief switching a large shock, compared to expectations can make beliefs switch. However a longer forward guidance might require a stronger commitment and a larger signal as the distance of the backward and forward states grow with the horizon. This sections studies how delivering forward guidance can create credibility. As seen before the size of the current period interest rate cut, and thus the size of the accommodation of the path is of crucial importance when backward-looking agents consider to believe the forward guidance.

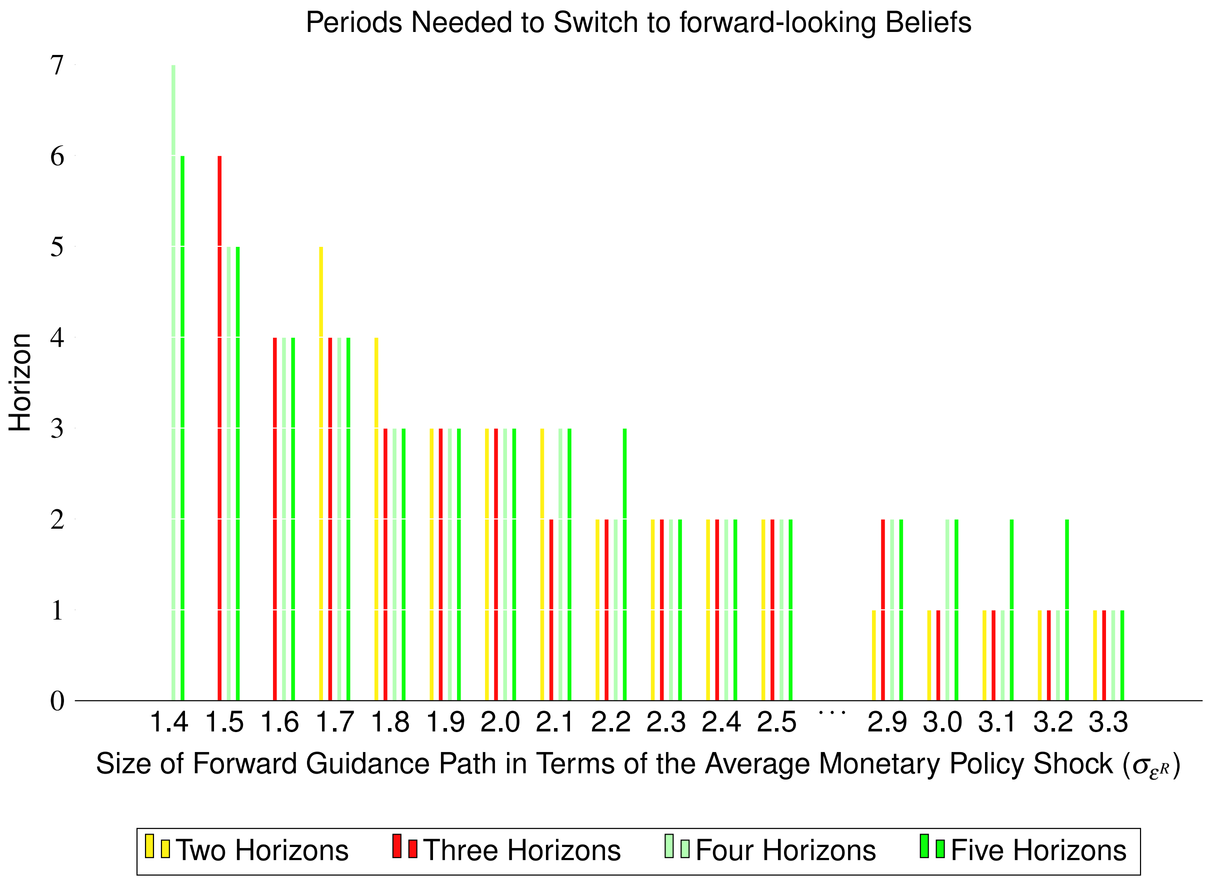
A larger accommodation today, measured in relation to the average size of the monetary policy shock, can make backward-looking beliefs switch to forward-looking. In other words, an agent facing a very large shock, will reconsider that it might be induced by not only the current monetary policy but by credible announcements of future policy action. Therefore the central bank by announcing a large interest cut today, and moving the whole economy by a lot, can signal to agents facing endogenous belief switching that the forward-looking beliefs are better suited to describe the path of the economy.

To study when such belief switching will take place consider that the central bank can decide to implement a forward guidance of any size and length. Agents start out having purely backward-looking beliefs, and are assumed to observe the economy as is, without measurement error. Under the initial beliefs the central bank has no credibility when announcing forward guidance, i.e. the puzzle does not emerge. Assume that the central bank would like to build a case for credible forward guidance and thus make beliefs forward-looking. This could be desirable as it creates another tool to control the business cycle. Future announcements of the interest rate can control of the economy better. What size of interest rate path is needed in order to make beliefs endogenously switch to being forward-looking? Does the credibility of forward guidance depend on its length? How long of a forward guidance, given size is needed to make beliefs forward-looking?

Figure 11 answers these questions. It shows the points where belief switching take place as a function of the length and the size of the forward guidance. The economy changes nature when the forward-looking beliefs become more likely and beliefs switch[28].

[28]As a reminder, the average size of monetary policy shock in the economy is of size 1. Similarly future forward guidance shocks have the same size. The forward-looking beliefs feature forward guidance up to 5 horizon ahead.

Figure 11: Central Bank Credibility - Period of Belief Switching



Notes: The chart shows the period when beliefs switch from backward to forward-looking as a function of the size and length of the forward guidance. backward-looking beliefs do not respond to future interest rate changes, while forward-looking beliefs are of a model with up-to five periods of forward guidance. The size of the path is measured in terms of the standard deviation of the average monetary policy shock. The yellow bar indicates two periods forward guidance, where the possibility of future forward guidance up to 5 horizons is considered. The red, light green and green show forward guidance of three, four and five horizons respectively.
(Source: Author's calculation)

A shorter forward guidance requires larger signal to switch beliefs. A two periods of forward guidance can make beliefs forward-looking if the size of the shock is large enough, i.e. larger than 1.7 times the average size of monetary accommodation. This is displayed on the chart 11 with yellow bar that shows that lower than 1.7 there is no switching. If the size of the monetary policy shock is two times the standard deviation of an average shock beliefs switch 3 periods after the announcement. For a 2.2 times shock it only requires 2 periods, while for a 3.0 times shock beliefs switch in the first period, i.e. similar to the case explained in Section 5.4.

On the other hand a longer forward guidance of five horizon, shown in dark green, requires smaller shocks for the switch to take place. Already a shock of 1.4 times the average monetary policy shock will make case for forward-looking beliefs, albeit only after the successful delivery of the promised path. If the size of the shock increases beliefs switch earlier. Looking at the chart one interesting pattern emerges.

Intuitively it is harder to believe a longer forward guidance than a shorter one. Endogenous belief switching delivers this result. The difference in credibility is due to the different size of variance different horizon forward guidance imply. A longer forward guidance commends a larger state dispersion, compared to a shorter, therefore it requires a larger shock to make it credible. For instance it takes a 3.3 times shock to make five horizon forward guidance credible in the first period, whereas for a four horizon forward guidance a shock of a size 3.1 is enough, for three horizon a shock of size 3.0 suffices.

Overall the figure shows that a larger forward guidance shock can make beliefs switch earlier, for any given length of the forward guidance. As we have seen, a larger shock represents a larger signal and thus can make beliefs switch endogenously earlier in the implementation of the path. Given the size of monetary policy accommodation, a longer forward guidance requires a longer delivery of the promised path to gain credibility.

However larger variance per se does not necessarily imply forward-looking behaviour. The relevant aspect is not the state dispersion, but the part of the variation that can be attributed to the signal. This ties endogenous belief switching to the signal extraction problem of the regime switching Kálmán filtering. Larger uncertainty can make filtering out the actual shock more difficult. So far we had the assumption, that the model is observed without errors, however in practice it is rarely the case that the model is a perfectly observed, and thus an accurate descriptor of the economy. The next section will illustrate how the signal to noise ratio will influence belief switching.

5.6 Noise and Signal

In the unconventional monetary policy literature the signalling effect of monetary policy is widely understood. With regards to learning this becomes even more important, as learning involves an estimation of the true model. The Switching Kálmán filter naturally gives rise to the signal extraction problem, in what follows I explain the relevance of the observation error and provide an intuition that generates practical interpretation of the noise and policy signal. The relation of noise to signal will play an important role in central bank policy transmission. If the central bank makes a too little action, gives a weak signal, agents confuse it with noise and will not respond to it. However should the signal be large compared to noise, the central bank can make beliefs become forward-looking.

When the agent filters the observables, he translates the DSGE solution to the observables space of the MSV using the observation equation:

$$y_t^{MSV} = \mathbf{H^i} y_t^i + \mathbf{u}_t, \tag{38}$$

where $\mathbf{H^i}$ is the so called observation equation matrix that selects the MSV states of the model. Denote $\mathbf{u}$ the measurement error with a covariance matrix that is $\mathbf{U}$[29]. The traditional interpretation of the observation error is measurement noise. It tells that there is an inherent imprecision to the observations that cannot be eliminated, and thus they have to be taken into account when designing a linear filter. I propose an econometric analogy of the measurement noise that is applicable here, and helps the intuition. Under general conditions one can decompose the total variance of any observable into conditional, explained variance and residual variation. Formally:

$$TSS = ESS + RSS, \tag{39}$$

where TSS stands for total variance, while ESS is the explained component of the variance, while RSS is the unexplained, residual variance. Then the fit of the model is the share of total variation explained, $R^2 = 1 - \frac{RSS}{TSS}$. The statistical name of fraction of variance unexplained

Consider that MSV states are the observables as in Equation 38. Think of the term $\mathbf{H^i} y_t^i$ as explanatory variables, that tells how much of the MSV total variation is due to central bank action. Another interpretation is the decomposition of the IRF, i.e. the movement of the states, to the explained and unexplained factors. Explained movements are those compatible with the model, be it a sequence of unanticipated monetary shocks, or a credible forward guidance. The unexplained, residual errors capture all sources of uncertainty that are beyond the scope of the model.

If $\mathbf{U}$ is small compared to the total state variation, then the changes in observables will be driven by the monetary policy, and a large fraction of it can be explained by the model. In other words the DSGE model has fit the data well.

While if observation errors are perceived to be large, it will result in the total variation being driven by the unexplained observation errors. The noise will be high compared to the signal, and thus the DSGE will not have a good fit to the data[30].

6 Endogenous Belief Switching in the Smets Wouters Model with Noise

In this section, we relax the assumption of perfect data fit in the DSGE model and explore how endogenously switching beliefs evolve in response to forward guidance when there is uncertainty in the data.

We present charts in Table 2 that show the impulse response functions to a forward guidance announcement of the SW07 model with endogenous belief switching. The charts vary based on the initial share of attentive beliefs and the fit of the model to the data.

When the signal-to-noise ratio is very small, representing high uncertainty, the model fit is

[29]The measurement error covariance is usually denoted by $\mathbf{R}$. In order not to confuse it with the mean squared error matrix of the beliefs, I changed the notation.

[30]In practice when estimating DSGE models, $\mathbf{U}$ is usually set to a very small number.

very low. In this case, agents do not learn much from the model, and their initial beliefs remain unchanged. The lack of information in the observables prevents belief updating and switching.

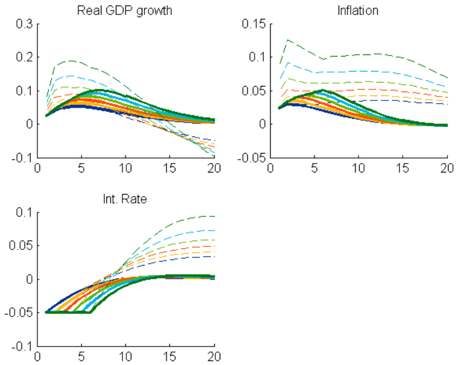
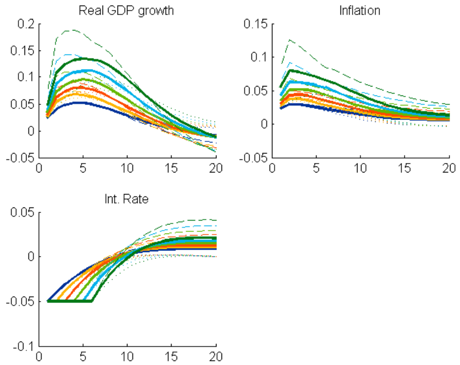
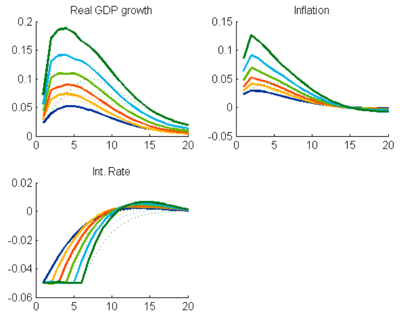
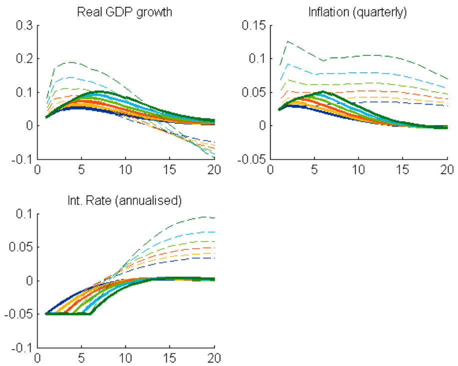
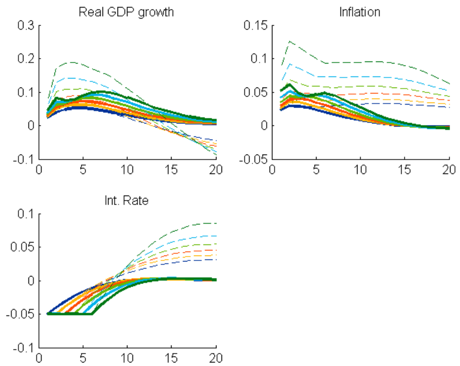
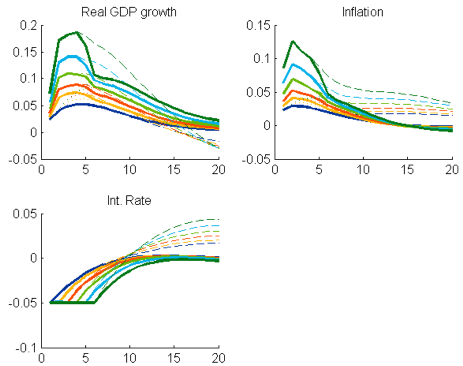
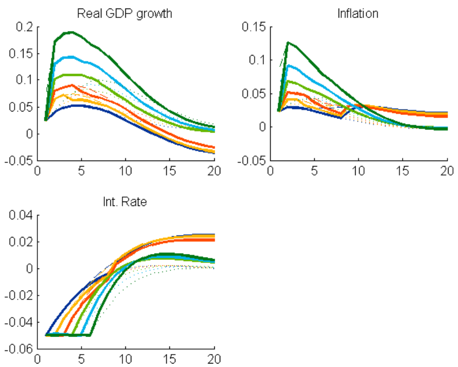
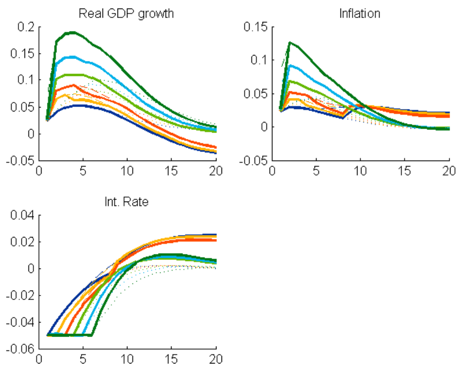
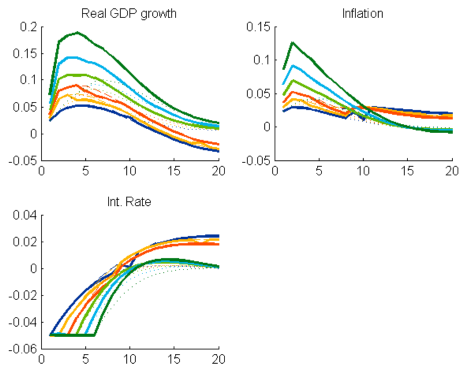
The forward guidance considered is a 6 horizon a time-dependent announcement, i.e. the central bank gives guidance about the short-term interest rate for 1, 2, ..., 6 periods ahead . The initial shares vary on the probability scale from 0% to 50% up until 100%. Where an a priori 0% probability of attentive beliefs tells that the initial beliefs are backward-looking. 50% represents a middle ground, where the agent is equally unsure about either being backward or forward-looking. Finally, 100% represents fully forward-looking belief initialization.

The model fit is measured as a fraction of variance explained by the model. The first row of the table represents the case when the signal-to-noise ratio very small. This is achieved by setting the observation error covariance matrix large in comparison to the long-run state variance. In this case the DSGE model of either regime has a very low fit, $\simeq 0$. In practice, this represents a highly uncertain environment, where the model does not tell anything about the data. If there is little information in the data, agents do not learn anything when filtering the data with the regime-switching Kálmán filter. In other words, if there is close to no information in the observables, then agents will not be able to update their a priori beliefs and will stick to their initial beliefs. This is shown in the charts as initial beliefs remain and no switching takes place.

The second row shows the case when the scale of the observation error compared to the state variance is similar in measure. A model fit of 50 $\%$ can be interpreted that half of the observed variation in the data is attributed to the observation error, i.e. unexplained variance. In this case, beliefs will converge on being backward-looking. This means that the forward guidance credibility cannot be maintained, and the central bank loses credibility. This scenario represents a high-uncertainty environment where expectations converge to the equilibrium of backward-looking beliefs.

Finally, when the observation error is low, representing a good fit of the model to the data, belief switching from backward to forward-looking beliefs is possible. In this case, the central bank can create credibility by giving a strong signal about its forward guidance commitment. The first panel in the last row, the case with 0% a priori forward-looking beliefs, illustrates how short forward guidance will lose credibility over the medium term, as the yellow and red ALMs switch back to backward-looking in the third and fourth periods respectively. These findings highlight the importance of data fit and uncertainty in shaping the effectiveness of forward guidance and the credibility of the central bank. A high degree of uncertainty can undermine the ability to establish forward-looking beliefs, while a good fit of the model can enhance credibility and make forward guidance more effective.

Table 2: Regime Switching Beliefs in SW07 with Adaptive Learning

Model fit	Initial a priori probability of forward-looking beliefs $Pr(S_0 = 1)$		
	0 %	50 %	100 %
$\simeq 0$ %			
50 %			
$\simeq 100$ %			

In summary, the credibility of forward guidance depends on the ability of the central bank to effectively signal its commitment. The results from Table 2 demonstrate several key points: Large unexplained observation errors in the economy hinder the ability of endogenous belief switching to occur. When there is little signal compared to the noise, belief updating and switching do not take place. Elevated uncertainty or imperfect fit of the DSGE model to the data diminishes the credibility of forward guidance. In these cases, backward-looking beliefs become the prevailing equilibrium expectations.

In a low uncertainty environment where the DSGE model provides a good description of the economy, endogenous belief switching from backward to forward-looking expectations can occur. This enables the central bank to gain credibility as agents dynamically learn to respond to a forward guidance.

6.1 Policy Implications

The results of this framework have important policy implications. It highlights the role of the central bank in the effectiveness of forward guidance and the challenges it may face in a liquidity trap.

On the one hand, if a central bank is conservative and fails to provide strong guidance, it may find itself trapped near the ZLB with no room to invoke a switch from backward to forward-looking beliefs. This can make it difficult for the central bank to escape the liquidity trap. Thus the ineffectiveness of forward guidance is a result of the liquidity trap.

On the other hand, forward guidance can be highly effective in low-uncertainty environments, but its effectiveness may diminish in high-uncertainty economies. Endogenous belief switching shows that expectations may become backward-looking if there is too much uncertainty compared to the signal provided by the central bank. In such an environment the central bank can only make forward guidance credible by activating other measures as well. Should it fail to do so, it might reinforce expectations to stay adaptive and become responsible for the inefficient impact of forward guidance announcements and the prevailing liquidity trap.

The case of Japan illustrates the challenges of escaping the ZLB. Endogenous belief switching helps explain why monetary policy in Japan failed to be effective. Perceived inaction by the central bank eroded the credibility of forward guidance, leading agents to become backward-looking. The heightened uncertainty in Japan further complicated the central bank's ability to provide a strong signal. Rethinking communication strategies and adopting unconventional measures could have been timely solutions to regain control of expectations and make forward guidance more effective.

The yield curve control implemented by the Bank of Japan can be seen as a step in the right direction. It acts as a stronger signaling device to anchor expectations to the forward guidance commitment and regain central bank control. By explaining its model to the public and embracing a wider range of unconventional measures, the central bank can reduce uncertainty, align its model with the economy, and turn agents' beliefs to be more forward-looking.

7 Conclusion

In conclusion, this paper introduces the concept of endogenous belief switching as a solution to the forward guidance puzzle in macroeconomics. By combining constant gain adaptive learning with regime switching Kálmán filtering, the paper highlights the role of central bank actions in shaping expectations and their credibility.

The findings show that constant gain adaptive learning can overcome the forward guidance puzzle when agents have backward-looking beliefs. However, if expectations are forward-looking and adaptive, the puzzle still persists. To address this, the paper introduces endogenous belief switching, which allows beliefs to dynamically switch between backward and forward-looking beliefs, endogenously learning to be Delphic or Odyssean in their beliefs of forward guidance.

The effectiveness of forward guidance is found to be endogenous: it is influenced not only by its size but also by the promised length of the guidance. A forward guidance with a large size and long horizon is more likely to make expectations switch to forward-looking. However, a short horizon forward guidance requires larger signals to avoid being misinterpreted as unanticipated shocks.

The paper emphasizes the importance of central bank credibility in shaping expectations. It shows that credibility can be built if the central bank provides a signal that is well understood by agents, either through a large and persistent delivery of promises or through communication, delivering a model that fits reality and dispels concerns of high uncertainty.

Endogenous belief switching offers new avenues for analyzing unconventional monetary policy and its impact on expectations. It provides a novel perspective on the conduct of monetary policy and opens up possibilities for further research in this area.

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Appendices

Switching Kálmán filter

In what follows, I will discuss the main steps needed to derive updating of the a priori state probabilities. Inference is filtering only - the probability distribution of a switch happening at time t depends only on past data, i.e. $1:t$. For a full discussion of the switching Kálmán filter please consult Murphy (1998).

Let us define a notation: y is the states, while y^{MSV} the observations. Denote the regimes with i.

Consider the state representation of the DSGE with observable noise on the MSV states:

$$y_t^i = \mathbf{F^i} y_{t-1}^i + \mathbf{w^i}_t, \tag{40}$$

$$y_t^{MSV} = \mathbf{H^i} y_t^i + \mathbf{u}_t \tag{41}$$

Where y_t^i is the state vector given belief i, either forward of backward-looking, $\mathbf{w^i}_t$ is the exogenous state disturbance. Denote its covariance matrix with Q_t^i. $\mathbf{H^i}$ is the emission matrix that selects the MSV states of the model. Furthermore $\mathbf{u}$ is the measurement noise with a covariance matrix that is usually denoted by $\mathbf{R}$. Not to confuse the mean squared error matrix under the beliefs, and I will express the measurement error covariance matrix with $\mathbf{U}$.

Furthermore one needs to specify an exogenous transition probability from state matrix Z $i \rightarrowtail j$. In the example I assume a highly persistent exogenous state transition probability of the form:

$$Z = \begin{pmatrix} 0.9999 & 0.0001 \\ 0.0001 & 0.9999 \end{pmatrix} \tag{42}$$

Introducing the notation:

$$y_{t|t}^{i(j)} = E\left[y_t | y_{1:t}^{MSV}, S_{t-1} = i, S_t = j\right] \tag{43}$$

Notice that the superscript in the brackets is the switching of regime from i to j in period t. The Equation 43 tells, what the value of the full state is given the (full) history of the MSV states, if it switches from regime i to j.

The switching Kálmán filter pass will be the following: First, the state distribution is inherited. It is all possible combination of states $y_{t|t-1}^{i(j)}$ and their respective covariance matrix based on information from $t-1$. The indices of states are looped over before progressing to the next step of the filter. The notation below exemplifies the filter as conditional on being in i switching to the next regime j. If the index is the same, then no regime switch takes place, if it is different it represents switching. As in the Kálmán filter the first step is called the prediction:

$$y^{i(j)}_{t|t-1} = \mathbf{F^j} y^i_{t-1}, \tag{44}$$
$$Q^{i(j)}_{t|t-1} = \mathbf{F^j} Q^i_{t-1} (\mathbf{F^j})' + Q^j_{t-1}. \tag{45}$$

Then, we compute the Kálmán gain given switching:

$$K^{i(j)} = Q^{i(j)}_{t|t-1} (\mathbf{H^j})' (\mathbf{H^j} Q^{i(j)}_{t|t-1} (\mathbf{H^j})' + \mathbf{U}) \tag{46}$$

Using the gain update the one can generate the nowcast, i.e. posterior of the state and state covariance matrix given information t:

$$y^{i(j)}_{t|t} = y^{i(j)}_{t|t-1} + K^{i(j)} (y^{MSV}_t - \mathbf{H^j} y^{i(j)}_{t|t-1}); \tag{47}$$
$$Q^{i(j)}_{t|t} = (I - K^{i(j)} \mathbf{H^j}) Q^{i(j)}_{t|t-1}; \tag{48}$$

With the nowcast, the likelihood of data given $S_t = j$ and $S_{t-1} = i$ can be computed that is the object of my application of the filter:

$$e^{i(j)}_t = y^{MSV}_t - \mathbf{H^j} y^{i(j)}_{t|t-1}, \tag{49}$$
$$L^{i(j)}_t = \sqrt{det(\mathbf{H^j} Q^{i(j)}_{t|t-1} \mathbf{H^j}' + \mathbf{U})} \cdot exp^{-\frac{1}{2}\Sigma\left(e^{i(j)}_t \left(\mathbf{H^j} Q^{i(j)}_{t|t-1} \mathbf{H^j}' + \mathbf{U}\right) e^{i(j)}_t\right)} \tag{50}$$

Finally one can update the a priori probabilities $Pr(S_t = i|t-1)$ using the following algorithm for all $i, j \in \{1,2\}$ and all t:

$$Pr(S_t = j|t, S_{t-1} = i) = \frac{L^{i(j)}_t Z(i,j) Pr(S_t = i|t-1)}{\sum_{i\in\{1,2\}} \sum_{j\in\{1,2\}} L^{i(j)}_t Z(i,j) Pr(S_t = i|t-1)} \tag{51}$$
$$Pr(S_t = j|t) = \sum_{i\in\{1,2\}} Pr(S_t = j|t, S_{t-1} = i) \tag{52}$$

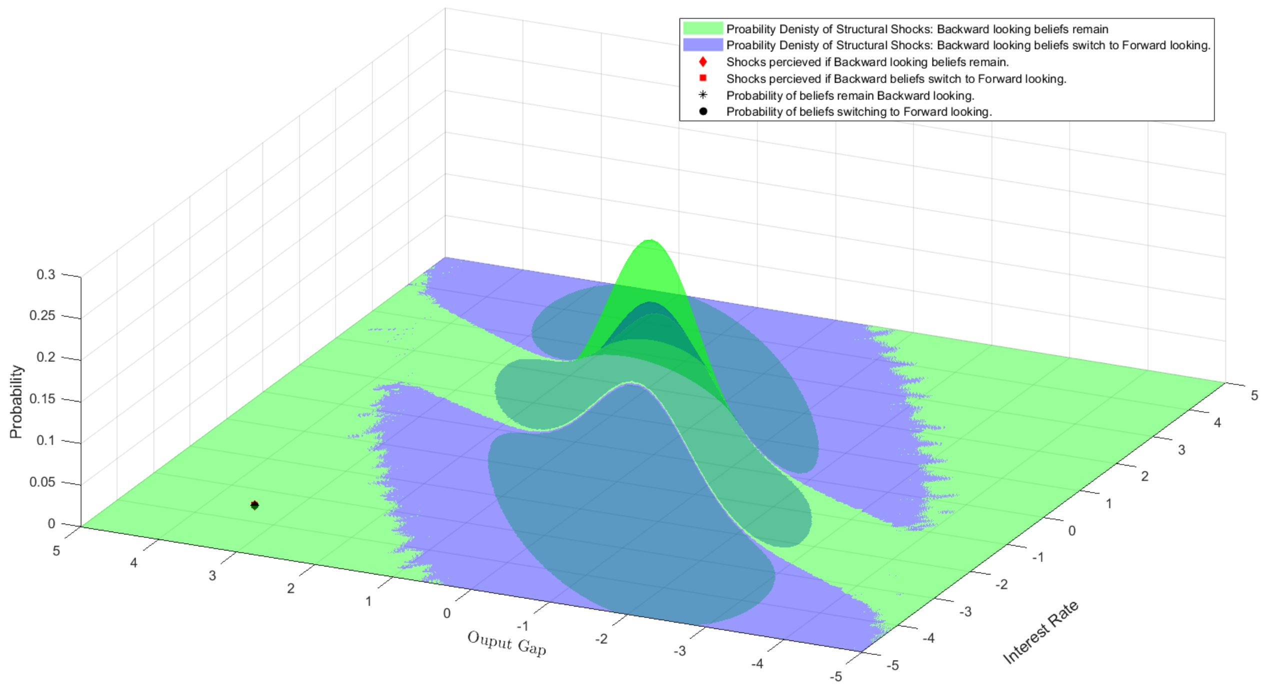
The final collapsing step assures that states are merged from across the regimes with the weighted probabilities:

$$y^j_{t|t} = \sum_{i\in\{1,2\}} y^{i(j)}_{t|t} \cdot \frac{Pr(S_t = j|t, S_{t-1} = i)}{Pr(S_t = j|t)}, \tag{53}$$
$$Q^j_{t|t} = \frac{Pr(S_t = j|t, S_{t-1} = i)}{Pr(S_t = j|t)} \left(Q^j_{t-1} + \left(y^{i(j)}_{t|t} - y^j_{t|t}\right) \left(y^{i(j)}_{t|t} - y^j_{t|t}\right)' \right). \tag{54}$$

There are two points needed to recognise. First, is the Markov assumption on the exogenous state state transition matrix Z, and the role it plays. It scales the likelihood and regulates switching. This is important as switching Kálmán filters have been documented to show instability of regimes and display way too many jumps. However over-regularizing the switching, and imposing an identity matrix, eliminates changing of regimes entirely. Therefore having a reasonable yet persistent exogenous regime dynamics is preferred. This is implemented with the calibration of entries in Z. Second, is the role of the observation error covariance matrix, $\mathbf{U}$. It is added to the (observation space compressed) state variance matrix, when computing the likelihood. That is the variation of the data is either driven by the model or the observation. It's relative size and possible correlation structure compared to that of the state variances is key in determining switching. I assume only the scale, expressed in terms of the average trace of the state covariance matrix, varies, the correlation structure does not.

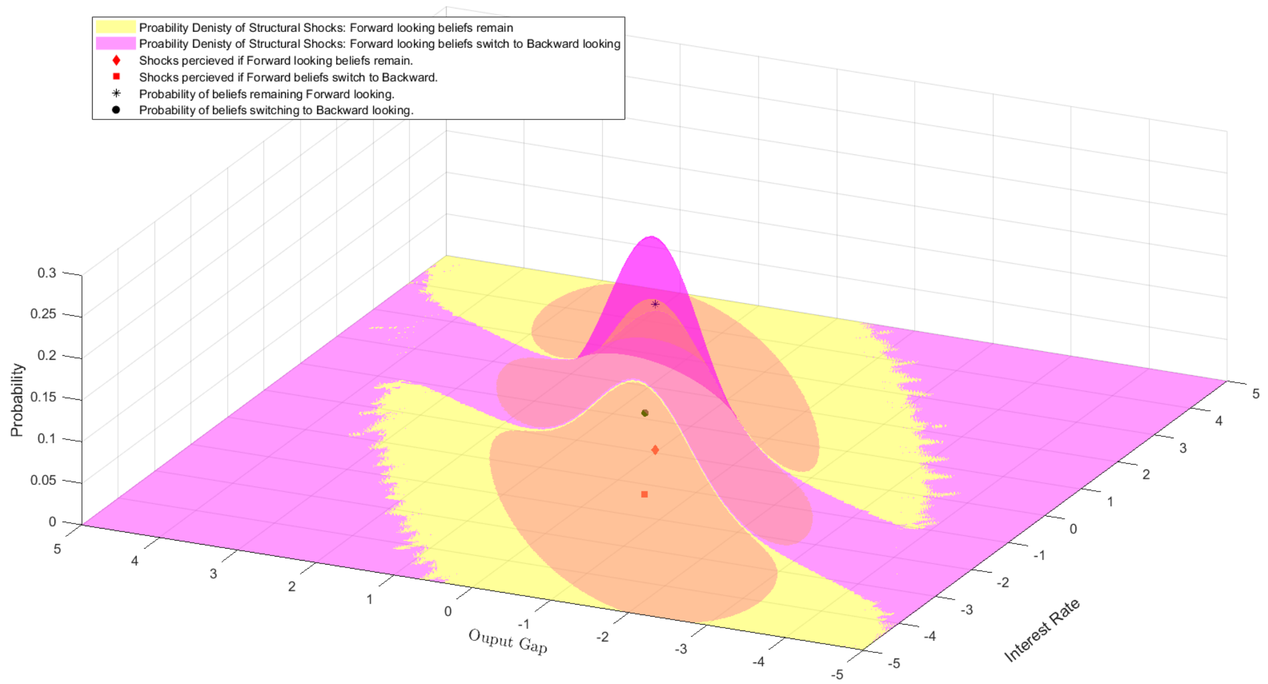
Additional Figures

Figure 12: Belief Switching Illustration - Bird's eye view - 1$^{\text{st}}$ Period



(Source: Author's calculation)

Figure 13: Belief Switching Illustration - Bird's eye view - 2nd Period



(Source: Author's calculation)

Dynamic equations

$$i_t = r_t 4 \tag{55}$$

$$\pi^{ann}{}_t = \pi^{ann}{}_t \tag{56}$$

$$\pi^q{}_t = 4\,\pi_t \tag{57}$$

$$y^{gap}{}_t = y_t - y^{flex}{}_t \tag{58}$$

$$y_t = y_t \tag{59}$$

$$g_t = \varepsilon^{gov}{}_t \tag{60}$$

$$\pi_{t-1_t} = \pi^q{}_{t-1} \tag{61}$$

$$\pi_{t-2_t} = \pi_{t-1_{t-1}} \tag{62}$$

$$\pi_{t-3_t} = \pi_{t-1_t} + \pi_{t-2_t} \tag{63}$$

$$i_t = \rho\, i_{t-1} + \theta_\pi \pi^q{}_t\, + \theta_x y_t - \sigma_r\, \varepsilon^R{}_t \tag{64}$$

$$g_t = coffispol\, \pi_{tt} \tag{65}$$

$$\varepsilon_{at} = \alpha\, r^{k,flex}{}_t + (1-\alpha)\, w^{flex}{}_t \tag{66}$$

$$z^{flex}{}_t = r^{k,flex}{}_t\, \frac{1}{\frac{\psi}{1-\psi}} \tag{67}$$

$$r^{k,flex}{}_t = w^{flex}{}_t + l^{flex}{}_t - k^{s,flex}{}_t \tag{68}$$

$$k^{s,flex}{}_t = z^{flex}{}_t + k^{flex}{}_{t-1} \tag{69}$$

$$i^{flex}{}_t = \frac{1}{1+\bar{\beta}\,\gamma_c} \left(i^{flex}{}_{t-1} + \bar{\beta}\,\gamma_c\, i^{flex}{}_{t+1} + \frac{1}{{\gamma_c}^2\,\varphi}\, q^{flex}{}_t \right) + \varepsilon^i{}_t \tag{70}$$

$$q^{flex}{}_t = \left(-r^{flex}{}_t\right) + c_2 * \varepsilon^b_{t\,t}\, \frac{1}{\frac{1-\frac{\lambda}{\gamma_c}}{\sigma_c\left(1+\frac{\lambda}{\gamma_c}\right)}} + \frac{\bar{r}^k}{\bar{r}^k+1-\delta}\, r^{k,flex}{}_{t+1} + \frac{1-\delta}{\bar{r}^k+1-\delta}\, q^{flex}{}_{t+1} \tag{71}$$

$$\begin{aligned} c^{flex}{}_t = {} & c_2 * \varepsilon^b_{t\,t} + \frac{\frac{\lambda}{\gamma_c}}{1+\frac{\lambda}{\gamma_c}}\, c^{flex}{}_{t-1} + \frac{1}{1+\frac{\lambda}{\gamma_c}}\, c^{flex}{}_{t+1} \\ & + \frac{(\sigma_c - 1)\; 1/\phi_w * (1-\alpha)/\alpha * \bar{r}^k * k_{ss}/y_{ss}}{\sigma_c\left(1+\frac{\lambda}{\gamma_c}\right)} \left(l^{flex}{}_t - l^{flex}{}_{t+1}\right) - r^{flex}{}_t\, \frac{1-\frac{\lambda}{\gamma_c}}{\sigma_c\left(1+\frac{\lambda}{\gamma_c}\right)} \end{aligned} \tag{72}$$

$$y^{flex}{}_t = c^{flex}{}_t\, c_{ss}/y_{ss} + i^{flex}{}_t\, i_{ss}/y_{ss} + \varepsilon^g{}_t + z^{flex}{}_t\, \bar{r}^k * k_{ss}/y_{ss} \tag{73}$$

$$y^{flex}{}_t = \phi_p \left(\varepsilon_{at} + \alpha\, k^{s,flex}{}_t + (1-\alpha)\, l^{flex}{}_t \right) \tag{74}$$

$$w^{flex}{}_t = l^{flex}{}_t\, \sigma_l + c^{flex}{}_t\, \frac{1}{1-\frac{\lambda}{\gamma_c}} - c^{flex}{}_{t-1}\, \frac{\frac{\lambda}{\gamma_c}}{1-\frac{\lambda}{\gamma_c}} \tag{75}$$

$$k^{flex}{}_t = k^{flex}{}_{t-1}\, (1 - \bar{i}/k) + i^{flex}{}_t\, \bar{i}/k + \varepsilon^i{}_t\, {\gamma_c}^2\, \varphi\, \bar{i}/k \tag{76}$$

$$\mu_{p_t} = \alpha\, r^k{}_t + (1-\alpha)\, w_t - \varepsilon_{at} \tag{77}$$

$$z_t = \frac{1}{\frac{\psi}{1-\psi}}\, r^k{}_t \tag{78}$$

$$r^k{}_t = w_t + l_t - k^s{}_t \tag{79}$$

$$k^s{}_t = z_t + k_{t-1} \tag{80}$$

$$i_t = \varepsilon^i{}_t + \frac{1}{1+\bar{\beta}\,\gamma_c}\left(i_{t-1} + \bar{\beta}\,\gamma_c\,i_{t+1} + \frac{1}{{\gamma_c}^2\,\varphi}\,q_t\right) \tag{81}$$

$$q_t = c_2 * \varepsilon^b_t{}_t\,\frac{1}{\frac{1-\frac{\lambda}{\gamma_c}}{\sigma_c\left(1+\frac{\lambda}{\gamma_c}\right)}} + (-r_t) + \pi_{t+1} + \frac{\bar{r}^k}{\bar{r}^k+1-\delta}\,r^k{}_{t+1} + \frac{1-\delta}{\bar{r}^k+1-\delta}\,q_{t+1} \tag{82}$$

$$\begin{aligned} c_t = {} & c_2 * \varepsilon^b_t{}_t + \frac{\frac{\lambda}{\gamma_c}}{1+\frac{\lambda}{\gamma_c}}\,c_{t-1} + \frac{1}{1+\frac{\lambda}{\gamma_c}}\,c_{t+1} \\ & + \frac{(\sigma_c-1)\;1/\phi_w*(1-\alpha)/\alpha*\bar{r}^k*k_{ss}/y_{ss}}{\sigma_c\left(1+\frac{\lambda}{\gamma_c}\right)}\,(l_t-l_{t+1}) - \frac{1-\frac{\lambda}{\gamma_c}}{\sigma_c\left(1+\frac{\lambda}{\gamma_c}\right)}\,(r_t-\pi_{t+1}) \end{aligned} \tag{83}$$

$$y_t = \varepsilon^g{}_t + c_{ss}/y_{ss}\,c_t + i_{ss}/y_{ss}\,i_t + \bar{r}^k*k_{ss}/y_{ss}\,z_t \tag{84}$$

$$y_t = \phi_p\left(\varepsilon_{at} + \alpha\,k^s{}_t + (1-\alpha)\,l_t\right) \tag{85}$$

$$\pi_t = \frac{1}{1+\bar{\beta}\,\gamma_c\,\iota_p}\left(\bar{\beta}\,\gamma_c\,\pi_{t+1} + \iota_p\,\pi_{t-1} + \mu_{p_t}\,\frac{\frac{(1-\xi_p)\left(1-\bar{\beta}\,\gamma_c\,\xi_p\right)}{\xi_p}}{1+(\phi_p-1)\,\varepsilon_p}\right) + \varepsilon^p{}_t \tag{86}$$

$$\begin{aligned} w_t = {} & \frac{1}{1+\bar{\beta}\,\gamma_c}\,w_{t-1} + \frac{\bar{\beta}\,\gamma_c}{1+\bar{\beta}\,\gamma_c}\,w_{t+1} + \pi_{t-1}\,\frac{\iota_w}{1+\bar{\beta}\,\gamma_c} - \pi_t\,\frac{1+\bar{\beta}\,\gamma_c\,\iota_w}{1+\bar{\beta}\,\gamma_c} + \pi_{t+1}\,\frac{\bar{\beta}\,\gamma_c}{1+\bar{\beta}\,\gamma_c} \\ & + \frac{(1-\xi_w)\left(1-\bar{\beta}\,\gamma_c\,\xi_w\right)}{\left(1+\bar{\beta}\,\gamma_c\right)\xi_w}\,\frac{1}{1+(\phi_w-1)\,\varepsilon_w}\left(\sigma_l\,l_t + \frac{1}{1-\frac{\lambda}{\gamma_c}}\,c_t - \frac{\frac{\lambda}{\gamma_c}}{1-\frac{\lambda}{\gamma_c}}\,c_{t-1} - w_t\right) + \varepsilon^w{}_t \end{aligned} \tag{87}$$

$$\varepsilon_{at} = \rho_a\,\varepsilon_{at-1} + \varepsilon^a{}_t \tag{88}$$

$$c_2 * \varepsilon^b_t{}_t = \rho_b\,c_2 * \varepsilon^b_t{}_{t-1} + \varepsilon^b{}_t \tag{89}$$

$$\varepsilon^g{}_t = \varepsilon^{gov}{}_t + \rho_g\,\varepsilon^g{}_{t-1} + \varepsilon^a{}_t\,\rho_{ga} \tag{90}$$

$$\varepsilon^i{}_t = \rho_i\,\varepsilon^i{}_{t-1} + \varepsilon^R{}_t \tag{91}$$

$$\varepsilon^r{}_t = \rho_r\,\varepsilon^r{}_{t-1} + \varepsilon^m{}_t \tag{92}$$

$$\varepsilon^p{}_t = \rho_p\,\varepsilon^p{}_{t-1} + \varepsilon^{p,aux}{}_t - \mu_p\,\varepsilon^{p,aux}{}_{t-1} \tag{93}$$

$$\varepsilon^{p,aux}{}_t = \varepsilon^p{}_t \tag{94}$$

$$\varepsilon^w{}_t = \rho_w\,\varepsilon^w{}_{t-1} + \varepsilon^{w,aux}{}_t - \mu_w\,\varepsilon^{w,aux}{}_{t-1} \tag{95}$$

$$\varepsilon^{w,aux}{}_t = \varepsilon^w{}_t \tag{96}$$

$$k_t = (1-\bar{i}/k)\,k_{t-1} + \bar{i}/k\,i_t + \varepsilon^i{}_t\,\varphi\,{\gamma_c}^2\,\bar{i}/k \tag{97}$$

$$\pi^{ann}{}_t = 0.25\left(\pi_{t-2_{t-1}} + \pi_{t-2_t} + \pi^q{}_t + \pi_{t-1_t}\right) \tag{98}$$

Baseline Smets Wouters Model variable declaration with adaptive learning

Table 3: Smets Wouters Model with Adaptive Learning: Endogenous variable definitions

Variable	LaTeX	Description
`ewma`	$\varepsilon^{w,aux}$	Auxiliary wage markup moving average variable
`epinfma`	$\varepsilon^{p,aux}$	Auxiliary price markup moving average variable
`zcapf`	z^{flex}	Capital utilization rate flex price economy
`rkf`	$r^{k,flex}$	rental rate of capital flex price economy
`kf`	$k^{s,flex}$	Capital services flex price economy
`pkf`	q^{flex}	real value of existing capital stock flex price economy
`cf`	c^{flex}	Consumption flex price economy
`invef`	i^{flex}	Investment flex price economy
`yf`	y^{flex}	Output flex price economy
`labf`	l^{flex}	hours worked flex price economy
`wf`	w^{flex}	real wage flex price economy
`rrf`	r^{flex}	real interest rate flex price economy
`mc`	μ_p	gross price markup
`zcap`	z	Capital utilization rate
`rk`	r^k	rental rate of capital
`k`	k^s	Capital services
`pk`	q	real value of existing capital stock
`c`	c	Consumption
`inve`	i	Investment
`y`	y	Output
`lab`	l	hours worked
`pinf`	π	Inflation
`w`	w	real wage
`r`	r	nominal interest rate
`a`	ε_a	productivity process
`b`	$c_2 * \varepsilon_t^b$	Scaled risk premium shock
`g`	ε^g	Exogenous spending
`qs`	ε^i	Investment-specific technology
`ms`	ε^r	Monetary policy shock process
`spinf`	ε^p	Price markup shock process
`sw`	ε^w	Wage markup shock process
`kpf`	k^{flex}	Capital stock flex price economy
`kp`	k	Capital stock
`pinf4`	π^{ann}	MMB: Annualized inflation

Table 3 – Continued

Variable	LaTeX	Description
eg	ε^{gov}	MMB: Government speding shock
interest	i	MMB: Common variable - Annualized nominal interest rate
inflation	π^{ann}	MMB: Common variable - Annualized inflation
inflationq	π^{q}	MMB: Common variable - Quarterly inflation
outputgap	y^{gap}	MMB: Common variable - Output gap[
output	y	MMB: Common variable - Output
fispol	g	MMB: Common variable - Government speding
inflationql	π_{t-1}	MMB: Common variable - Lagged Inflation
inflationql2	π_{t-2}	MMB: Common variable - 2x Lagged Inflation
inflationqls	π_{t-3}	MMB: Common variable - 3x Lagged Inflation

Table 4: Smets Wouters Model with Adaptive Learning: Exogenous variable definitions

Variable	LaTeX	Description
ea	ε^{a}	productivity shock
eb	ε^{b}	Investment-specific technology shock
eqs	ε^{R}	Investment-specific technology shock
em	ε^{m}	Monetary policy shock
epinf	ε^{p}	Price markup shock
ew	ε^{w}	Wage markup shock
interest_	ε^{R}	MMB: Common variable - MP shock
fiscal_	π_t	MMB: Common variable - Fiscal shock

Table 5: Smets Wouters Model with Adaptive Learning: Parameter Definitions

Variable	LaTeX	Description
cofintintb1	ρ	Taylor rule interest rate smoothing
cofintinf0	θ_π	Taylor rule inflation feedback
cofintoutp	θ_x	Taylor rule ouput gap feedback
std_r_	σ_r	Taylor rule monetary policy shock size (set to 0.01)
curvw	ε_w	Curvature Kimball aggregator wages
cgy	ρ_{ga}	Feedback technology on exogenous spending
curvp	ε_p	Curvature Kimball aggregator prices

Table 5 – Continued

Variable	LaTeX	Description
constelab	$\bar{l}$	steady state hours
constepinf	$\bar{\pi}$	steady state inflation rate
constebeta	$100(\beta^{-1}-1)$	time preference rate in percent
cmaw	μ_w	coefficient on MA term wage markup
cmap	μ_p	coefficient on MA term price markup
calfa	α	capital share
czcap	ψ	capacity utilization cost
csadjcost	φ	investment adjustment cost
ctou	δ	depreciation rate
csigma	σ_c	risk aversion
chabb	λ	external habit degree
cfc	ϕ_p	fixed cost share
cindw	ι_w	Indexation to past wages
cprobw	ξ_w	Calvo parameter wages
cindp	ι_p	Indexation to past prices
cprobp	ξ_p	Calvo parameter prices
csigl	σ_l	Frisch elasticity
clandaw	ϕ_w	Gross markup wages
crpi	r_π	Taylor rule inflation feedback
crdy	$r_{\Delta y}$	Taylor rule output growth feedback
cry	r_y	Taylor rule output level feedback
crr	ρ	interest rate persistence
crhoa	ρ_a	persistence productivity shock
crhob	ρ_b	persistence risk premium shock
crhog	ρ_g	persistence spending shock
crhoqs	ρ_i	persistence risk premium shock
crhoms	ρ_r	persistence monetary policy shock
crhopinf	ρ_p	persistence price markup shock
crhow	ρ_w	persistence wage markup shock
ctrend	$\bar{\gamma}$	net growth rate in percent
cg	$\frac{\bar{g}}{\bar{y}}$	steady state exogenous spending share
cgamma	γ_c	BGP growth rate of quarterly GDP
clandap	$\phi_{p,ss}$	SS fixed cost share
cbetabar	$\bar{\beta}$	SS SDF
cr	$\bar{r}*$	SS Real rate
cpie	$\bar{\pi}$	SS inflation rate
crk	$\bar{r}^k$	SS rental rate of capital

Table 5 – Continued

Variable	LaTeX	Description
`cw`	$\bar{w}$	SS real wage
`cikbar`	$\bar{i}/k$	SS investment rate
`cik`	$\bar{i}_{ss}/k_{ss}$	SS investment rate
`clk`	i_{ss}/k_{ss}	BGP detrended SS investment rate
`cky`	k_{ss}/y_{ss}	SS capital-output ratio
`ciy`	i_{ss}/y_{ss}	SS investment-output ratio
`ccy`	c_{ss}/y_{ss}	SS consumption-output ratio
`crkky`	$\bar{r}^k * k_{ss}/y_{ss}$	SS capital income share of output
`cwhlc`	$1/\phi_w * (1-\alpha)/\alpha * \bar{r}^k * k_{ss}/y_{ss}$	SS wage income share of output
`cwly`	$1-\bar{r}^k * k_{ss}/y_{ss}$	SS wage share
`conster`	$(\bar{r} * -1) * 100$	SS r* in percentage points

Smets Wouters Parameters calibration

Table 6: Smets Wouters Model with Adaptive Learning: Parameter Values

Parameter	Value	Description
ρ	0.900	Interest rate smoothing
θ_π	0.150	Inflation response
θ_x	0.013	Output gap response
σ_r	0.01	MP shock size (normalized to 0.01)
ε_w	10.000	Curvature Kimball aggregator wages
ρ_{ga}	0.519	Feedback technology on exogenous spending
ε_p	10.000	Curvature Kimball aggregator prices
$\bar{l}$	0.551	steady state hours
$\bar{\pi}$	0.787	steady state inflation rate
$100(\beta^{-1}-1)$	0.166	time preference rate in percent
μ_w	0.850	coefficient on MA term wage markup
μ_p	0.701	coefficient on MA term price markup
α	0.190	capital share
ψ	0.546	capacity utilization cost
φ	5.761	investment adjustment cost
δ	0.025	depreciation rate
σ_c	1.381	risk aversion
λ	0.713	external habit degree

Table 6 – Continued

Parameter	Value	Description
ϕ_p	1.606	fixed cost share
ι_w	0.585	Indexation to past wages
ξ_w	0.706	Calvo parameter wages
ι_p	0.243	Indexation to past prices
ξ_p	0.652	Calvo parameter prices
σ_l	1.838	Frisch elasticity
ϕ_w	1.500	Gross markup wages
r_π	2.044	Taylor rule inflation feedback
$r_{\Delta y}$	0.225	Taylor rule output growth feedback
r_y	0.088	Taylor rule output level feedback
ρ	0.810	interest rate persistence
ρ_a	0.958	persistence productivity shock
ρ_b	0.219	persistence risk premium shock
ρ_g	0.977	persistence spending shock
ρ_i	0.711	persistence risk premium shock
ρ_r	0.148	persistence monetary policy shock
ρ_p	0.889	persistence price markup shock
ρ_w	0.969	persistence wage markup shock
$\bar{\gamma}$	0.431	net growth rate in percent
$\frac{\bar{g}}{\bar{y}}$	0.180	steady state exogenous spending share
γ_c	1.004	BGP growth rate based on quarterly trend growth rate to GDP
$\phi_{p,ss}$	1.606	SS fixed cost share
$\bar{\beta}$	0.992	SS SDF
$\bar{r}*$	1.016	SS Real rate
$\bar{\pi}$	1.008	SS inflation rate
$\bar{r}^k$	0.033	SS rental rate of capital
$\bar{w}$	0.682	SS real wage
$\bar{i}/k$	0.029	SS investment rate
$\bar{i}_{ss}/k_{ss}$	0.029	SS investment rate
i_{ss}/k_{ss}	0.204	BGP detrended SS investment rate
k_{ss}/y_{ss}	5.827	SS capital-output ratio
i_{ss}/y_{ss}	0.171	SS investment-output ratio
c_{ss}/y_{ss}	0.649	SS consumption-output ratio
$\bar{r}^k * k_{ss}/y_{ss}$	0.190	SS capital income share of output
$1/\phi_w * (1-\alpha)/\alpha * \bar{r}^k * k_{ss}/y_{ss}$	0.832	SS wage income share of output
$1 - \bar{r}^k * k_{ss}/y_{ss}$	0.810	SS wage share

Table 6 – Continued

Parameter	Value	Description
$(\bar{r}*-1)*100$	1.555	SS r* in percentage points